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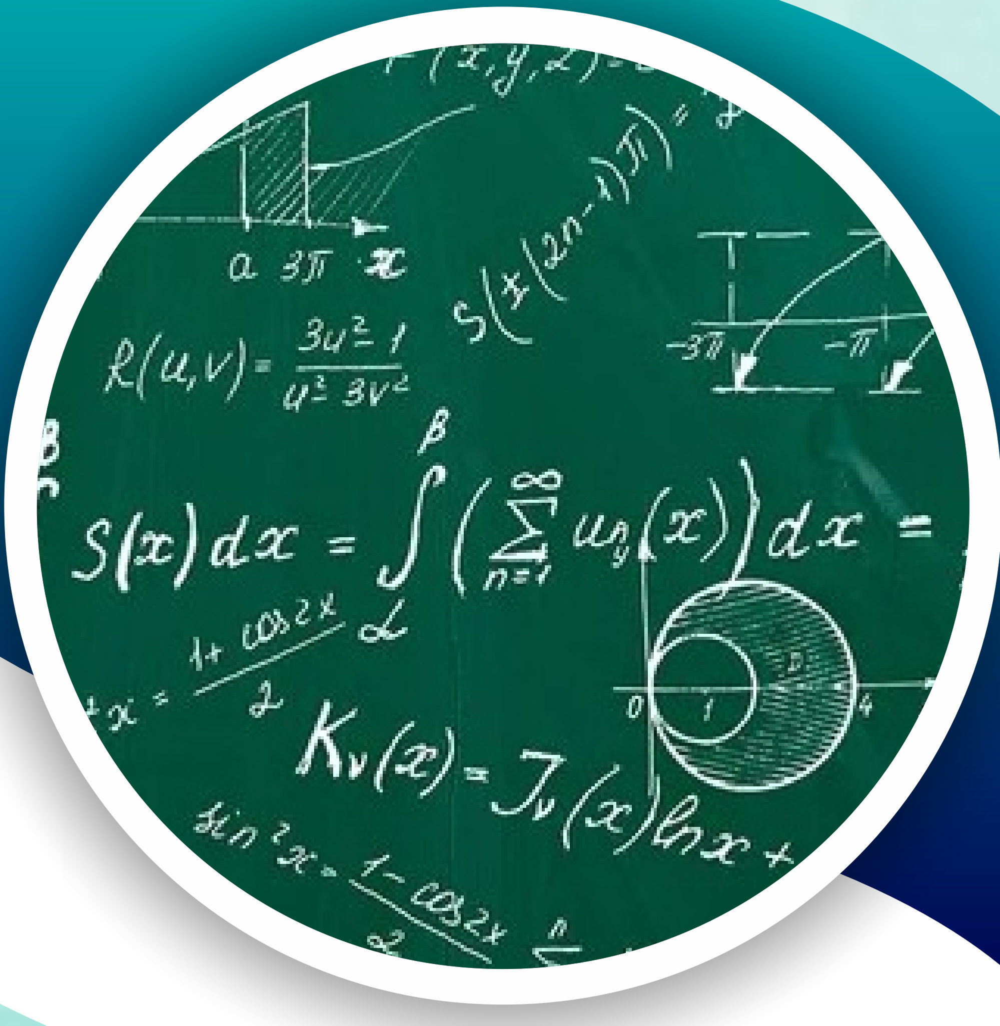
Discrete Mathematics
BSCCS-104



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Discrete Mathematics

Dr. Babasaheb Ambedkar Open University

Course Writer

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Discrete Mathematics

UNIT 1 : SETS

UNIT STRUCTURE

1.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- describe sets and their representations
- identify empty set, finite and infinite sets
- define subsets, super sets, power sets, universal set
- describe the use of Venn diagram for geometrical description of sets

- illustrate the set operations of union, intersection, difference and complement
- know the different algebraic laws of set-operations
- illustrate the application of sets in solving practical problems.

1.2 INTRODUCTION

One of the widely used concepts in present day Mathematics is the concept of Sets. It is considered the language of modern Mathematics. The whole structure of Pure or Abstract Mathematics is based on the concept of sets. German mathematician **Georg Cantor** (1845-1918) developed the theory of sets and subsequently many branches of modern Mathematics have been developed based on this theory. In this unit, preliminary concepts of sets, set operations and some ideas on its practical utility will be introduced.

1.3 SETS AND THEIR REPRESENTATION

A set is a collection of well-defined objects. By well-defined, it is meant that given a particular collection of objects as a set and a particular object, it must be possible to determine whether that particular object is a member of the set or not.

The objects forming a set may be of any sort– they may or may not have any common property. Let us consider the following collections :

i) the collection of the prime numbers less than 15 i.e., 2, 3, 5, 7, 11, 13
ii) the collection of 0, a, Sachin Tendulkar, the river Brahmaputra
iii) the collection of the beautiful cities of India
iv) the collection of great mathematicians.

Clearly the objects in the collections (i) and (ii) are well-defined. For example, 7 is a member of (i), but 20 is not a member of (i). Similarly, ‘a’ is a member of (ii), but M. S. Dhoni is not a member. So, the collections (i) and (ii) are sets. But the collections (iii) and (iv) are not sets, since the objects in these collections are not well-defined.

The objects forming a set are called **elements** or **members** of the set. Sets are usually denoted by capital letters A, B, C, ...; X, Y, Z, ..., etc., and the elements are denoted by small letters a, b, c, ...; x, y, z, ..., etc. If 'a' is an element of a set A, then we write $a \in A$ which is read as 'a belongs to the set A' or in short, 'a belongs to A'. If 'a' is not an element of A, we write $a \notin A$ and we read as 'a does not belong to A'. For example, let A be the set of prime number less than 15.

Then $2 \in A, 3 \in A, 5 \in A, 7 \in A, 11 \in A, 14 \in A$

$1 \notin A, 4 \notin A, 17 \notin A$, etc.

Representation of Sets : Sets are represented in the following two methods :

1. Roster or tabular method
2. Set-builder or Rule method

In the Roster method, the elements of a set are listed in any order, separated by commas and are enclosed within braces, For example,

A = {2, 3, 5, 7, 11, 13}

B = {0, a, Sachin Tendulcar, the river Brahmaputra}

C = {1, 3, 5, 7, ...}

In the set C, the elements are all the odd natural numbers. We cannot list all the elements and hence the dots have been used showing that the list continues indefinitely.

In the Rule method, a variable x is used to represent the elements of a set, where the elements satisfy a definite property, say P(x). Symbolically, the set is denoted by {x : P(x)} or {x | p(x)}. For example,

A = {x : x is an odd natural number}

B = $\{x : x^2 - 3x + 2 = 0\}$, etc.

If we write these two sets in the Roster method, we get,

A = {1, 3, 5, ...}

B = {1, 2}

Some Standard Symbols for Sets and Numbers : The following standard symbols are used to represent different sets of numbers :

N = {1, 2, 3, 4, 5, ...}, the set of natural numbers

Z = {..., –3, –2, –1, 0, 1, 2, 3, ...}, the set of integers

Q = $\{x : x = p/q;\ p, q \in Z, q \neq 0\}$, the set of rational numbers

R = {x : x is a real number}, the set of real numbers

NOTE

1) It should be noted that the symbol ':' of '|' stands for the phrase 'such that'.
2) While writing a set in Roster method, only distinct elements are listed. For example, if A is the set of the letters of the word MATHEMATICS, then we write
   A = {A, E, C, M, H, T, S, I}
   The elements may be listed in any order.

Z^+, Q^+, R^+ respectively represent the sets of positive integers, positive rational numbers and positive real numbers. Similarly Z^-, Q^-, R^- represent respectively the sets of negative integers, negative rational numbers and negative real numbers. Z^0, Q^0, R^0 represent the sets of non-zero integers, non-zero rational numbers and non-zero real numbers.

Illustrative Examples :

1. Examine which of the following collections are sets and which are not :
   i) the vowels of the English alphabet
   ii) the divisors of 56
   iii) the brilliant students degree-course of Guwahati
   iv) the renowned cricketers of Assam.

Solution :

i) It is a set, V = {a, e, i, o, u}
ii) It is a set, D = {1, 2, 4, 7, 8, 14, 28, 56}
iii) not a set, elements are not well-defined.
iv) not a set, elements are not well-defined.

2. Write the following sets in Roster method :
   i) the set of even natural numbers less than 10
   ii) the set of the roots of the equation $x^2–5x+6 = 0$
   iii) the set of the letters of the word EXAMINATION

Solution :

i) {2, 4, 6, 8}
ii) {2, 3}
iii) {E, X, A, M, I, N, T, O}

3. Write the following sets in Rule method :
   i) E = {2, 4, 6, ...}
   ii) A = {2, 4, 8, 16, 32}
   iii) B = {1, 8, 27, 64, 125, 216}

Solution :

i) $E = \{x : x = 2n, n \in N\}$
ii) $A = \{x : x = 2^n, n \in N, n < 6\}$
iii) $B = \{x : x = n^3, n \in N, n \leq 6\}$

CHECK YOUR PROGRESS

Q.1. Express the following sets in Roster method :

i) A = {x : x is a day of the week}

ii) B = {x : x is a month of the year}

iii) C = {x : $x^3 - 1 = 0$}

iv) D = {x : x is a positive divisor of 100}

v) E = {x : x is a letter of the word ALGEBRA}

Q.2. Express the following sets in Set-builder method :

i) A = {January, March, May, July, August, October, December}

ii) B = {0, 3, 8, 15, 24, ...}

iii) C = {0, ±5, ±10, ±15, ...}

iv) D = {a, b, c, ..., x, y, z}

Q.3. Write true or false :

i) $5 \in N$ ii) $\frac{1}{2} \in Z$ iii) $-1 \in Q$

iv) $\sqrt{2} \in R$ v) $\sqrt{-1} \in R$ vi) $-3 \notin N$

1.4 THE EMPTY SET

Definition : A set which does not contain any element is called an **empty set** or a **null set** or a **void set**. It is denoted by ϕ.

The following sets are some examples of empty sets.

i) the set {x : $x^2 = 3$ and $x \in Q$}

ii) the set of people in Assam who are older than 500 years

iii) the set of real roots of the equation $x^2 + 4 = 0$

iv) the set of Lady President of India born in Assam.

1.5 FINITE AND INFINITE SETS

Let us consider the sets

A = {1, 2, 3, 4, 5}

and B = {1, 4, 7, 10, 13, ...}

If we count the members (all distinct) of these sets, then the counting process comes to an end for the elements of set A, whereas for the elements of B, the counting process does not come to an end. In the first case we say that A is a finite set and in the second case, B is called an infinite set. A has finite number of elements and number of elements in B are infinite.

Definition : A set containing finite number of distinct elements so that the process of counting the elements comes to an end after a definite stage is called a **finite set**; otherwise, a set is called an **infinite set**.

NOTE

A finite set can always be expressed in roster method. But an infinite set cannot be always expressed in roster method as the elements may not follow a definite pattern. For example, the set of real numbers, R cannot be expressed in roster method.

Example : State which of the following sets are finite and which are infinite.

i) the set of natural numbers N
ii) the set of male persons of Assam as on January 1, 2009.
iii) the set of prime numbers less than 20
iv) the set of concentric circles in a plane
v) the set of rivers on the earth.

Solution :

i) N = {1, 2, 3, ...} is an infinite set
ii) it is a finite set
iii) {2, 3, 5, 7, 11, 13, 17, 19} is a finite set
iv) it is an infinite set
v) it is a finite set.

1.6 EQUAL SETS

Definition : Two sets A and B are said to be equal sets if every element of A is an element of B and every element of B is also an element of A. In otherwords, A is equal to B, denoted by A = B if A and B have exactly the same elements. If A and B are not equal, we write $A \neq B$.

Let us consider the sets

$A = \{1, 2\}$

$B = \{x : (x-1)(x-2) = 0\}$

$C = \{x : (x-1)(x-2)(x-3) = 0\}$

Clearly $B = \{1, 2\}$, $C = \{1, 2, 3\}$ and hence $A = B$, $A \neq C$, $B \neq C$.

Example : Find the equal and unequal sets :

i) A = {1, 4, 9}

ii) B = {1^2, 2^2, 3^3}

iii) C = {x : x is a letter of the word TEAM}

iv) D = {x : x is a letter of the word MEAT}

v) E = {1, {4}, 9}

Solution : A = B, C = D, A ≠ C, A ≠ D, A ≠ E, B ≠ C, B ≠ D, B ≠ E, C ≠ E, D ≠ E

> **NOTE**
> According to equality of sets discussed above, the sets
> A = {1, 2, 3} and
> B = {1, 2, 2, 2, 3, 1, 3}
> are equal, since every member of A is a member of B and also every member of B is a member of A. This is why identical elements are taken once only while writing a set in the Roster method.

1.7 SUBSETS, SUPERSETS, PROPER SUBSETS

Let us consider the sets A = {1, 2, 3}, B = {1, 2, 3, 4} and C = {3, 2, 1}. Clearly, every element of A is an element of B, but A is not equal to B. Again, every element of A is an element of C, and also A is equal to C. In both cases, we say that A is a subset of B and C. In particular, we say that A is a proper subset of B, but A is not a proper subset of C.

Definition : If every element of a set A is also an element of another set B, then A is called a **subset** of B, or A is said to be contained in B, and is denoted by $A \subseteq B$. Equivalently, we say that B contains A or B is a **superset** of A and is denoted by $B \supseteq A$. Symbolically, $A \subseteq B$ means that for all x, if $x \in A$ then $x \in B$.

If A is a subset of B, but there exists atleast one element in B which is not in A, then A is called a **proper subset** of B, denoted by $A \subset B$. In otherwords, $A \subset B \Leftrightarrow (A \subseteq B \text{ and } A \neq B)$.

The symbol '⇔' stands for 'logically implies and is implied by' (see unit 5).

Some examples of proper subsets are as follows :

$N \subset Z$, $N \subset Q$, $N \subset R$,

$Z \subset Q$, $Z \subset R$, $Q \subset R$.

It should be noted that any set A is a subset of itself, that is, $A \subseteq A$. Also, the null set ϕ is a subset of every set, that is, $\phi \subseteq A$ for any set A. Because, if $\phi \subseteq A$, then there must exist an element $x \in \phi$ such that $x \notin A$. But $x \notin \phi$, hence we must accept that $\phi \subseteq A$.

Combining the definitions of equality of sets and that of subsets, we get $A = B \Leftrightarrow (A \subseteq B \text{ and } B \subseteq A)$

Illustrative Examples :

1. Write true or false :

i) $1 \subset \{1, 2, 3\}$

ii) $\{1, 2\} \subseteq \{1, 2, 3\}$

iii) $\phi \subseteq \{\{\phi\}\}$

iv) $\phi \subseteq \{\phi, \{1\}, \{a\}\}$

v) $\{a, \{b\}, c, d\} \subset \{a, b, \{c\}, d\}$

Solution :

i) False, since $1 \in \{1, 2, 3\}$.

ii) True, since every element of {1, 2} is an element of {1, 2, 3}.

iii) False, since ϕ is not an element of $\{\{\phi\}\}$.

iv) True, since ϕ is subset of every set.

v) False, since $\{b\} \notin \{a, b, \{c\}, d\}$ and $c \notin \{a, b, \{c\}, d\}$.

1.8 POWER SET

Let us consider a set A = {a, b}. A question automatically comes to our mind– 'What are the subsets of A?' The subsets of A are ϕ, {a}, {b} and A itself.

These subsets, taken as elements, again form a set. Such a set is called the power set of the given set A.

Definition : The set consisting of all the subsets of a given set A as its elements, is called the **power set** of A and is denoted by P(A) or 2^A.

Thus, P(A) or $2^A = \{X : X \subseteq A\}$

Clearly,

i) $P(\phi) = \{\phi\}$

ii) if A = {1}, then $P^A = \{\phi, \{1\}\}$

iii) if A = {1, 2}, then $P^A = \{\phi, \{1\}, \{2\}, A\}$

iv) if A = {1, 2, 3}, then $P^A = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, A\}$

From these examples we can conclude that if a set A has n elements, then P(A) has 2^n elements.

1.9 UNIVERSAL SET

A set is called a **Universal Set** or the **Universal discourse** if it contains all the sets under consideration in a particular discussion. A universal set is denoted by U.

Example :

i) For the sets {1, 2, 3}, {3, 7, 8}, {4, 5, 6, 9}

We can take U = {1, 2, 3, 4, 5, 6, 7, 8, 9}

ii) In connection with the sets N, Z, Q we can take R as the universal set.

iii) In connection with the population census in India, the set of all people in India is the universal set, etc.

CHECK YOUR PROGRESS

Q.4. Find the empty sets, finite and infinite sets :

i) the set of numbers divisible by zero

ii) the set of positive integers less than 15 and divisible by 17

iii) the set of planets of the solar system

iv) the set of positive integers divisible by 4

v) the set of coplanar triangles

vi) the set of Olympians from Assam participating in 2016, Rio Olympics.

Q.5. Examine the equality of the following sets :

i) A = {2, 3}, B = $\{x : x^2-5x+6 = 0\}$

ii) A = {x : x is a letter of the word WOLF}

B = {x : x is a letter of the word FLOW}

iii) A = {a, b, c}, B = {a, {b, c}}

Q.6. Write true or false :

i) $\{1, 3, 5\} \subseteq \{5, 1, 3\}$ ii) $\{a\} \subset \{\{a\}, b\}$

iii) $\{x : (x-1)(x-2) = 0\} \not\subset \{x : (x^2-3x+2)(x-3) = 0\}$

Q.7. Write down the power sets of the following sets :

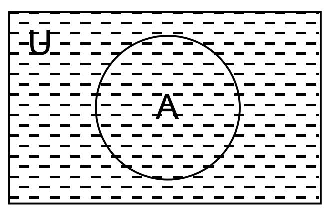
i) A = {1, 2, 3, 4} ii) B = {1, {2, 3}}

Q.8. Give examples to show that (A $\subseteq$ B and B $\subseteq$ C) $\Rightarrow$ A $\subseteq$ C.

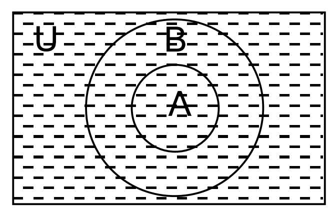
1.10 VENN DIAGRAM

Simple plane geometrical areas are used to represent relationships between sets in meaningful and illustrative ways. These diagrams are called **Venn-Euler** diagrams, or simply the **Venn-diagrams**.

In Venn diagrams, the universal set U is generally represented by a set of points in a rectangular area and the subsets are represented by circular regions within the rectangle, or by any closed curve within the rectangle. As an illustration Venn diagrams of A $\subset$ U, A $\subset$ B $\subset$ U are given below :



A $\subset$ U



A $\subset$ B $\subset$ U

Similar Venn diagrams will be used in subsequent discussions illustrating different algebraic operations on sets.

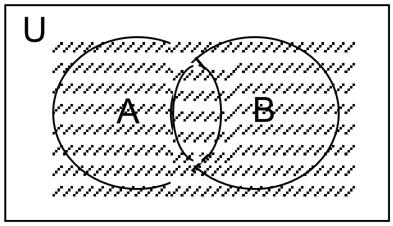
1.11 SET OPERATIONS

We know that given a pair of numbers x and y, we can get new numbers x + y, x – y, xy, $^x/_y$ (with y $\neq$ 0) under the operations of addition, subtraction, multiplication and division. Similarly, given the two sets A and B we can form new sets under set operations of **union**, **intersection**, **difference** and **complements**. We will now define these set operations, and the new sets thus obtained will be shown with the help of Venn diagrams.

1.11.1 Union of Sets

Definition : The union of two sets A and B is the set of all elements which are members of set A or set B or both. It is denoted by $A \cup B$, read as 'A union B' where '$\cup$' is the symbol for the operation of 'union'. Symbolically we can describe $A \cup B$ as follows :

$A \cup B = \{x : x \in A \text{ or } x \in B\}$



$A \cup B$ (Shaded)

It is obvious that $A \subseteq A \cup B$, $B \subseteq A \cup B$

Example 1 : Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 5, 6\}$

Then $A \cup B = \{1, 2, 3, 4, 5, 6\}$

Example 2 : Let Q be the set of all rational numbers and K be the set of all irrational numbers and R be the set of all real numbers.

Then $Q \cup K = R$

Identities : If A, B, C be any three sets, then

i) $A \cup B = B \cup A$

ii) $A \cup A = A$

iii) $A \cup \phi = A$

iv) $A \cup U = U$

v) $(A \cup B) \cup C = A \cup (B \cup C)$

Proof :

i) $A \cup B = \{x : x \in A \text{ or } x \in B\}$
$= \{x : x \in B \text{ or } x \in A\}$
$= B \cup A$

ii) $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$

iii) $A \cup \phi = \{x : x \in A \text{ or } x \in \phi\} = \{x : x \in A\} = A$

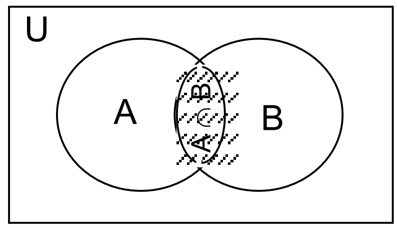
iv) $A \cup U = \{x : x \in A \text{ or } x \in U\}$
$= \{x : x \in U\}$, since $A \subset U$
$= U$

v) $(A \cup B) \cup C = \{x : x \in A \cup B \text{ or } x \in C\}$
$= \{x : (x \in A \text{ or } x \in B) \text{ or } x \in C\}$
$= \{x : x \in A \text{ or } (x \in B \text{ or } x \in C\}$
$= \{x : x \in A \text{ or } x \in B \cup C\}$
$= A \cup (B \cup C)$

1.11.2 Intersection of Sets

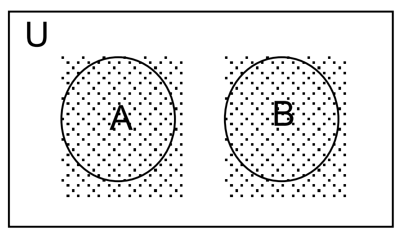
Definition : The intersection of two sets A and B is the set of all elements which are members of both A and B. It is denoted by $A \cap B$, read as 'A intersections B', where '$\cap$' is the symbol for the operation of 'intersection'. Symbolically we can describe it as follows:

$A \cap B = \{x : x \in A \text{ and } x \in B\}$



$A \cap B$ (Shaded)

From definition it is clear that if A and B have no common element, then $A \cap B = \phi$. In this case, the two sets A and B are called **disjoint sets**.



$A \cap B = \phi$

It is obvious that $A \cap B \subseteq A$, $A \cap B \subseteq B$.

Example 1 : Let A = {a, b, c, d}, B = {b, d, 4, 5}

Then $A \cap B$ = {b, d}

Example 2 : Let A = {1, 2, 3}, B = {4, 5, 6}

Then $A \cap B = \phi$.

Identities :

i) $A \cap B = B \cap A$

ii) $A \cap A = A$

iii) $A \cap \phi = \phi$

iv) $A \cap U = A$

v) $(A \cap B) \cap C = A \cap (B \cap C)$

vi) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof :

i) $A \cap B = \{x : x \in A \text{ and } x \in B\}$

$= \{x : x \in B \text{ and } x \in A\}$

$= B \cap A$

ii) $A \cap A = \{x : x \in A \text{ and } x \in A\}$

$= \{x : x \in A\}$

$= A$

iii) Since ϕ has no element, so A and ϕ have no common element.

Hence $A \cap \phi = \phi$

iv) $A \cap U = \{x : x \in A \text{ and } x \in U\}$

$= \{x : x \in A\}$, since $A \subset U$

$= A$

v) $(A \cap B) \cap C = \{x : x \in A \cap B \text{ and } x \in C\}$

$= \{x : (x \in A \text{ and } x \in B) \text{ and } x \in C\}$

$= \{x : x \in A \text{ and } (x \in B \text{ and } x \in C\}$

$= \{x : x \in A \text{ and } x \in B \cap C\}$

$= A \cap (B \cap C)$

vi) $x \in A \cap (B \cup C) \Leftrightarrow x \in A \text{ and } x \in (B \cup C)$

$\Leftrightarrow x \in A \text{ and } (x \in B \text{ or } x \in C)$

$\Leftrightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)$

$\Leftrightarrow x \in (A \cap B) \text{ or } x \in (A \cap C)$

$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$

So, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Similarly, it can be proved that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

NOTE

$x \in A \cap B$

$\Rightarrow x \in A \text{ and } x \in B$

But, $x \notin A \cap B$

$\Rightarrow x \notin A \text{ or } x \notin B$

Again, $x \in A \cup B$

$\Rightarrow x \in A \text{ or } x \in B$

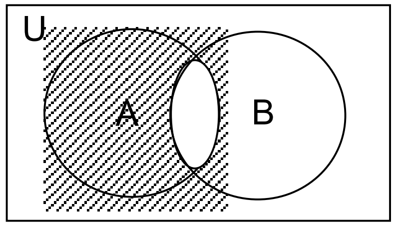
But, $x \notin A \cup B$

$\Rightarrow x \notin A \text{ and } x \notin B$

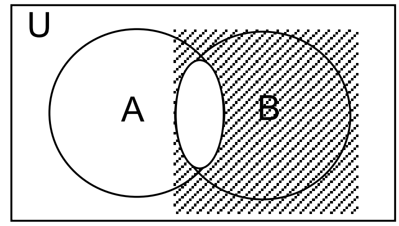
1.11.3 Difference of Sets

Definition : The difference of two sets A and B is the set of all elements which are members of A, but not of B. It is denoted by A–B. Symbolically, $A - B = \{x : x \in A \text{ and } x \notin B\}$

Similarly, $B - A = \{x : x \in B \text{ and } x \notin A\}$



A – B (Shaded)



B – A (Shaded)

Example : Let A = {1, 2, 3, 4, 5}, B = {1, 4, 5}, C = {6, 7, 8}

Then A – B = {2, 3}

A – C = A

B – C = B

B – A = ϕ

Properties :

i) $A - A = \phi$

ii) $A - B \subseteq A$, $B - A \subseteq B$

iii) $A - B$, $A \cap B$, $B - A$ are mutually disjoint and

$(A - B) \cup (A \cap B) \cup (B - A) = A \cup B$

iv) $A - (B \cup C) = (A - B) \cap (A - C)$

v) $A - (B \cap C) = (A - B) \cup (A - C)$

Proof : We prove (iv), others are left as exercises.

$x \in A - (B \cup C) \Leftrightarrow x \in A$ and $x \notin (B \cup C)$

$\Leftrightarrow x \in A$ and ($x \notin B$ and $x \notin C$)

$\Leftrightarrow$ ($x \in A$ and $x \notin B$) and ($x \in A$ and $x \notin C$)

$\Leftrightarrow x \in (A - B)$ and $x \in (A - C)$

$\Leftrightarrow x \in (A - B) \cap (A - C)$

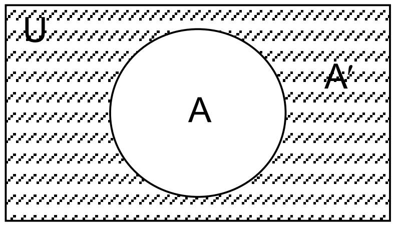
So, $A - (B \cup C) \subseteq (A - B) \cap (A - C)$, $(A - B) \cap (A - C) \subseteq A - (B \cup C)$

Hence, $A - (B \cup C) = (A - B) \cap (A - C)$.

1.11.4 Complement of a Set

Definition : If U be the universal set of a set A, then the set of all those elements in U which are not members of A is called the **Compliment** of A, denoted by A^C or A'.

Symbolically, $A' = \{x : x \in U \text{ and } x \notin A\}$.



A′ (Shaded)

Clearly, $A' = U - A$.

Example : Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9} and A = {2, 4, 6, 8}

Then A′ = {1, 3, 5, 7, 9}

Identities : i) $U' = \phi$, $\phi' = U$

ii) $(A')' = A$

iii) $A \cup A' = U$, $A \cap A' = \phi$

iv) $A - B = A \cap B'$, $B - A = B \cap A'$

v) $(A \cup B)' = A' \cap B'$, $(A \cap B)' = A' \cup B'$

NOTE

The identities $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$ are know as **DeMorgan's Laws**.

Proof : We prove $(A \cup B)' = A' \cap B'$. The rest are left as exercises.

$$(A \cup B)' = \{x : x \in U \text{ and } x \notin A \cup B\}$$
$$= \{x : x \in U \text{ and } (x \notin A \text{ and } x \notin B\}$$
$$= \{x : (x \in U \text{ and } x \notin A) \text{ and } (x \in U \text{ and } x \notin B)\}$$
$$= \{x : x \in A' \text{ and } x \in B'\}$$
$$= A' \cap B'.$$

Illustrative Examples :

1. If U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

A = {2, 4, 6, 8, 10}

B = {3, 6, 9}

and C = {1, 2, 3, 4, 5}, then find

(i) $A \cup B$, (ii) $A \cap C$, (iii) $B \cap C$, (iv) A', (v) $A \cup B'$, (vi) $C' \cap B$,

(vii) $A' \cup C'$, (viii) $A - C$, (ix) $A - (B \cup C)'$, (x) $A' \cap B'$.

Solution : i) $A \cup B$ = (2, 3, 4, 6, 8, 9, 10}

ii) $A \cap C$ = {2, 4}

iii) $B \cap C$ = {3}

iv) A' = {1, 3, 5, 7, 9}

v) B' = {1, 2, 4, 5, 7, 8, 10}

So, $A \cup B'$ = {1, 2, 4, 5, 6, 7, 8, 10}

vi) $C' = \{6, 7, 8, 9, 10\}$

So, $C' \cap B = \{6, 9\}$

vii) From (iv) & (vi), $A' \cup C' = \{1, 3, 5, 6, 7, 8, 9, 10\}$

viii) $A - C = \{6, 8, 10\}$

ix) $B \cup C = \{1, 2, 3, 4, 5, 6, 9\}$

$(B \cup C)' = \{7, 8, 10\}$

So, $A - (B \cup C)' = \{2, 4, 6\}$

x) From (iv) & (v), $A' \cap B' = \{1, 5, 7\}$.

2. Verify the identities :

i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

ii) $(A \cup B)' = A' \cap B'$

taking $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $C = \{3, 4, 5\}$ and $U = \{1, 2, 3, 4, 5, 6\}$.

Solution : i) $B \cap C = \{3, 4\}$

$A \cup (B \cap C) = \{1, 2, 3, 4\}$ ......(1)

$A \cup B = \{1, 2, 3, 4\}$, $A \cup C = \{1, 2, 3, 4, 5\}$

$(A \cup B) \cap (A \cup C) = \{1, 2, 3, 4\}$ ......(2)

From (1) & (2), we get

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

ii) $A' = \{4, 5, 6\}$, $B' = \{1, 5, 6\}$

$A' \cap B' = \{5, 6\}$ ......(3)

$(A \cup B)' = \{1, 2, 3, 4\}' = \{5, 6\}$ ......(4)

From (3) & (4), wet get

$(A \cup B)' = A' \cap B'$.

CHECK YOUR PROGRESS

Q.9. Find the following sets :

i) $\phi \cap \{\phi\}$ ii) $\{\phi\} \cap \{\phi\}$

iii) $\{\phi, \{\phi\}\} - \{\phi\}$ iv) $\{\phi, \{\phi\}\} - \{\{\phi\}\}$

Q.10. If $A = \{a, b, c\}$, $B = \{c, d, e\}$, $U = \{a, b, c, d, e, f\}$ then find

i) $A \cup B$ ii) $A \cap B$ iii) $A - B$ iv) $B - A$ v) A'

Q.11. If A = {3n : n $\in$ N}, B = {n : n $\in$ N, n < 20} then find $A \cap B$, $B - A$.

Q.12. If A = {x : x $\in$ R, $1 < x \leq 7$}, B = { x : x $\in$ R, $3 \leq x < 12$} then find $A \cup B$, $A \cap B$, $A - B$, $B - A$.

Q.13. Using numerical examples, verify that :

i) $A - B = B' - A'$

ii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap C)$

iii) $A - (B \cap C) = (A - B) \cup (A - C)$

Q.14. Prove that : i) $B - A \subseteq A'$ ii) $B - A' = B \cap A$

iii) $A \cup B = \phi \Rightarrow A = \phi$ and $B = \phi$.

1.12 LAWS OF THE ALGEBRA OF SETS

In the preceding discussions we have stated and proved various identities under the operations of union, intersection and complement of sets. These identities are considered as **Laws of Algebra of Sets**. These laws can be directly used to prove different propositions on Set Theory. These laws are given below :

1. **Idempotent laws :** $A \cup A = A$, $A \cap A = A$
2. **Commutative laws :** $A \cup B = B \cup A$, $A \cap B = B \cap A$
3. **Associative laws :** $A \cup (B \cup C) = (A \cup B) \cup C$,
   $A \cap (B \cap C) = (A \cap B) \cap C$
4. **Distributive laws :** $A \cup (B \cap C) = (A \cup B) \cap (A \cap C)$,
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. **Identity laws :** $A \cup \phi = A$, $A \cup U = U$
   $A \cap U = A$, $A \cap \phi = \phi$
6. **Complement laws :** $A \cup A' = U$, $A \cap A' = \phi$
   $(A')' = A$, $U' = \phi$, $\phi' = U$
7. **De Morgan's laws :** $(A \cup B)' = A' \cap B'$
   $(A \cap B)' = A' \cup B'$.

Let us illustrate the application of the laws in the following examples :

Example 1 : Prove that $A \cap (A \cup B) = A$

Solution : $A \cap (A \cup B) = (A \cup \phi) \cap (A \cup B)$, using identity law

$= A \cup (\phi \cap B)$, using distributive law

$= A \cup (B \cap \phi)$, using commutative law

$= A \cup \phi$, using identity law

$= A$, again using identity law

Example 2 : Prove that $A \cap (A' \cup B) = A \cap B$

Solution : $A \cap (A' \cup B) = (A \cap A') \cup (A \cap B)$, using distributive law

$= \phi \cup (A \cap B)$, using complement law

$= (A \cap B) \cup \phi$, using commutative law

$= A \cap B$, using identity law

1.13 TOTAL NUMBER OF ELEMENTS IN UNION OF SETS IN TERMS OF ELEMENTS IN INDIVIDUAL SETS AND THEIR INTERSECTIONS

We shall now prove a theorem on the total number of elements in the union of two sets in terms of the number of elements of the two individual sets and the number of elements in their intersection. Its application in solving some practical problems concerning everyday life will be shown in the illustrative examples.

Theorem : If A and B are any two finite sets,

then $|A \cup B| = |A| + |B| - |A \cap B|$

[The symbol |S| represents total number of elements in a set S]

Proof : Let $|A| = n$, $|B| = m$, $|A \cap B| = k$

Then from the Venn diagram,

we get $|A - B| = n - k$, $|B - A| = m - k$

We know that $A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$

Where $A - B$, $A \cap B$, $B - A$ are mutually disjoint.

Hence $|A \cup B| = |A - B| + |A \cap B| + |B - A|$

$= (n - k) + k + (m - k)$

$= n + m - k$

$= |A| + |B| - |A \cap B|$

Deduction :

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

Proof : $|A \cup B \cup C| = |(A \cup B) \cup C|$

$= |A \cup B| + |C| - |(A \cup B) \cap C|$

$= |A| + |B| - |A \cap B| + |C| - |(A \cap C) \cup (B \cap C)|$

$= |A| + |B| + |C| - |A \cap B| -$

$[|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|]$

$= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| +$

$|A \cap B \cap C|$

Corollaries :

i) If A and B are disjoint, then $A \cap B = \phi$ and so $|A \cap B| = 0$.

Hence $|A \cup B| = |A| + |B|$,

which is known as the **Sum Rule of Counting**.

ii) If A, B and C are mutually disjoint, then as above

$|A \cup B \cup C| = |A| + |B| + |C|$.

Illustrative Examples :

1. In a class of 80 students, everybody can speak either English or Assamese or both. If 39 can speak English, 62 can speak Assamese, how many can speak both the languages?

Solution : Let A, B be the sets of students speaking English and Assamese respectively.

Then $|A \cup B| = 80$, $|A| = 39$, $|B| = 62$.

We are to find $|A \cap B|$.

Now $|A \cup B| = |A| + |B| - |A \cap B|$

So, $|A \cap B| = |A| + |B| - |A \cup B| = 39 + 62 - 80 = 21$.

Hence, 21 students can speak both the languages.

2. Among 60 students in a class, 28 got class I in SEM I and 31 got class I in SEM II. If 20 students did not get class I in either SEMESTERS, how many students got class I in both the SEMESTERS?

Solution : Let A and B be the sets of students who got class I in SEM I and SEM II respectively.

So, $|A| = 28$, $|B| = 31$.

20 students did not get class I in either SEMESTERS out of 60 students in the class.

Hence $|A \cup B| = 60 - 20 = 40$

But $|A \cup B| = |A| + |B| - |A \cap B|$

i.e., $40 = 28 + 31 - |A \cap B|$ So, $|A \cap B| = 19$

Therefore, 19 students did not get class I is both the SEMESTERS.

3. Out of 200 students, 70 play cricket, 60 play football, 25 play hockey, 30 play both cricket and football, 22 play both cricket and hockey, 17 play both football and hockey and 12 play all the three games. How many students do not play any one of the three games?

Solution : Let C, F, H be the sets of students playing cricket, football and hockey respectively. Then

$|C| = 70$, $|F| = 60$, $|H| = 25$,

$|C \cap F| = 30$, $|C \cap H| = 22$, $|F \cap H| = 17$, $|C \cap F \cap H| = 12$.

$$\text{So, } |C \cup F \cup H| = |C| + |F| + |H| - |C \cap F| - |C \cap H| - |F \cap H| + |C \cap F \cap H|$$

$$= 70 + 60 + 25 - 30 - 22 - 17 + 12 = 98$$

Thus 98 students play atleast one of the three games.

Hence, number of students not playing any one of the three games $= 200 - 98 = 102$.

EXERCISES

1. Write 'true' or 'false' with proper justification :
   - i) the set of even prime numbers is an empty set
   - ii) $\{x : x+2 = 5, x < 0\}$ is an empty set
   - iii) $\{3\} \subset \{1, 2, 3\}$
   - iv) $x \in \{\{x, y\}\}$
   - v) $\{a, b\} \subseteq \{a, b, \{c\}\}$
   - vi) if A be any set, then $\phi \subseteq A \subseteq U$
2. Which of the following sets are equal?

   $A = \{x : x^2 + x - 2 = 0\}$

B $= \{x : x^2 - 3x + 2 = 0\}$

C $= \{x : x \in Z, |x| = 1\}$

D $= \{-2, 1\}$

E $= \{1, 2\}$

F $= \{x : x^2 - 1 = 0\}$

3. Find the sets which are finite and which are infinite :
   i) the set of natural numbers which are multiple of 7
   ii) the set of all districts of Assam
   iii) the set of real numbers between 0 and 1
   iv) the set of lions in the world.
4. If $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{0, 2, 3, 6\}$, $B = \{1, 2, 6, 8\}$, $C = \{3, 7, 8, 9\}$, then find A', B', C', $(A \cup B) \cap C$, $(A \cup B') \cup C'$, $(A \cap B) \cap C'$, $(A - C) \cup B'$, $(B - A)' \cap C$.
5. If $A \cup B = B$ and $A \cap B = B$, then what is the relation between A and B?
6. Verify the following identities with numerical examples :
   i) $A - B = B' - A'$
   ii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
   iii) $A - (B \cap C) = (A - B) \cup (A - C)$
   iv) $A - (B \cup C) = (A - B) \cap (A - C)$.
7. Write down the power set of the set $A = \{\{\phi\}, a, \{b, c\}\}$.
8. Given $A = \{\{a, b\}, \{c\}, \{d, e, f\}\}$, how many elements are there in $P(A)$?
9. Using numerical examples, show that
   i) $(A \cap B) \cup (A - B) = A$
   ii) $A \cup B = A \cup (B - A)$
   iii) $A \cup B = B \cup (A - B)$
   iv) $B - A \subseteq A'$
   v) $B - A' = B \cap A$.
10. Using Venn diagrams show that
    i) $A \cup B \subset A \cup C$ but $B \not\subset C$
    ii) $A \cap B \subset A \cap C$ but $B \not\subset C$
    iii) $A \cup B = A \cup C$ but $B \neq C$.

11. Give numerical examples for the results given in 10.
12. Show that (A – B) – C = (A – C) – (B – C).
13. Out of 100 persons, 45 drink tea and 35 drink coffee. If 10 persons drink both, how many drink neither tea nor coffee?
14. Using sets, find the total number of integers from 1 to 300 which are not divisible by 3, 5 and 7.
15. 90 students in a class appeared in tests for Physics, Chemistry and Mathematics. If 55 passed in Physics, 45 passed in Chemistry, 60 passed in Mathematics, 40 both in Physics and Chemistry, 30 both in Chemistry and Mathematics, 35 both in Physics and Mathematics and 20 passed in all the three subjects, then find the number of students failing in all the three subjects.

1.14 LET US SUM UP

- A set is a collection of well-defined and distinct objects. The objects are called members or elements of the set.
- Sets are represented by capital letters and elements by small letters. If 'a' is an element of set A, we write $a \in A$, otherwise $a \notin A$.
- Sets are represented by (i) Roster or Tabular method and (ii) Rule or Set-builder method.
- A set having no element is called **empty set** or **null set** or **void set**, denoted by ϕ.
- A set having a finite number of elements is called a **finite set**, otherwise it is called an **infinite set**.
- Two sets A and B are equal, i.e. A = B if and only if every element of A is an element of B and also every element of B is an element of A, otherwise $A \neq B$.
- A is a **subset** of B, denoted by $A \subseteq B$ if every element of A is an element of B and A is a **proper subset** of B if $A \subseteq B$ and $A \neq B$. In this case, we write $A \subset B$.
- A = B if and only if $A \subseteq B$ and $B \subseteq A$.

- The set of all the subsets 8 a set A is called the **power set** of A, denoted by P(A) or 2^A. If $|A| = n$, then $|P(A)| = 2^n$.
- Venn diagrams are plane geometrical diagrams used for representing relationships between sets.
- The union of two sets A and B is $A \cup B$ which consists of all elements which are either in A or B or in both. $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- The intersection of two sets A and B is $A \cap B$ which consists of all the elements common to both A and B.
- For any two sets A and B, the difference set, A – B consists of all elements which are in A, but not in B. $A - B = \{x : x \in A \text{ and } x \notin B\}$
- The **Universal set U** is that set which contains all the sets under any particular discussion as its subsets.
- The **complement** of a set A, denoted by A^c or A' is that set which consists of all those elements in U which are not in A.
  $A' = \{x : x \in U \text{ and } x \notin A\} = U - A$
- Following are the **Laws of Algebra of Sets** :
  $A \cup A = A,\ A \cap A = A$
  $A \cup B = B \cup A,\ A \cap B = B \cap A$
  $A \cup (B \cup C) = (A \cup B) \cup C,\ A \cap (B \cap C) = (A \cap B) \cap C$
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C),\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  $A \cup \phi = A,\ A \cup U = U,\ A \cap U = A,\ A \cap \phi = \phi$
  $A \cup A' = U,\ A \cap A' = \phi,\ (A')' = A,\ U' = \phi,\ \phi' = U$
  $(A \cup B)' = A' \cap B',\ (A \cap B)' = A' \cup B'$.
- $|A \cup B| = |A| + |B| - |A \cap B|$
  $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

1.15 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 : i) A = {Monday, Tuesday, Wednessday, Thursday, Friday, Saturday, Sunday}

ii) B = {January, February, March, April, May, June, July, August, September, October, November, December}

iii) $C = \{1, w, w^2\}$

iv) D = {1, 2, 4, 5, 10, 20, 25, 50, 100}

v) E = {A, B, E, G, L, R}

Ans. to Q. No. 2 : i) A = {x : x is a month of the year having 31 days}

ii) $B = \{x : x = n^2 - 1, n \in N\}$

iii) $C = \{x : x = 5n, n \in Z\}$

iv) D = {x : x is a letter of the English Alphabet}

Ans. to Q. No. 3 : i) True, ii) False, iii) True, iv) True, v) False, vi) True.

Ans. to Q. No. 4 : i) ϕ, ii) ϕ, iii) finite, iv) infinite, v) infinite, vi) ϕ.

Ans. to Q. No. 5 : i) B = {2, 3} = A

ii) A = {W, O, L, F}, B = {F, L, O, W} and so, A = B

iii) $A \neq B$; since $b \in A$ but $b \notin B$.

Ans. to Q. No. 6 : i) True

ii) False, since $\{a\} \in \{\{a\}, b\}$

iii) $\{x : (x-1)(x-2) = 0\} = \{1, 2\}$, $\{x : (x^2-3x+2)(x-3) = 0\} = \{1, 2, 3\}$

Hence $\{x : (x-1)(x-2) = 0 \subset \{x : (x^2-3x+2)(x-3) = 0\}$ and so, the given result is false.

Ans. to Q. No. 7 : i) $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, A\}$

ii) $P(B) = \{\phi, \{1\}, \{\{2, 3\}\}, B\}$

Ans. to Q. No. 8 : Let A = {1, 2}, B = {0, 1, 2, 3}, C = {0, 1, 2, 3, 4, 5, 7}

Ans. to Q. No. 9 : i) ϕ, ii) $\{\phi\}$, iii) $\{\{\phi\}\}$, iv) $\{\phi\}$

Ans. to Q. No. 10 : i) $A \cap B = \{a, b, c, d, e\}$; ii) {C}; iii) A – B = {a, b}, iv) B – A = {d, e}; v) A′ = {d, e, f}

Ans. to Q. No. 11 : A = {3, 6, 9, 12, 15, 18, 21, ...},

B = {1, 2, 3, ..., 18, 19, 20},

Hence $A \cap B = \{3, 6, 9, 12, 15, 18\}$

and B – A = {1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19}

Ans. to Q. No. 12 : $A \cup B = \{x : 1 < x < 12, x \in R\}$

$A \cap B = \{x : x \in R, 3 \leq x \leq 7\}$

$A - B = \{x : x \in R, 1 < x < 3\}$

$B - A = \{x : x \in R, 7 < x < 12\}$

Ans. to Q. No. 13 : Take U = {p, q, r, s, t, u, v, w, x, y, z}

$A = \{p, q, u, v, x, y\}$

$B = \{q, v, y, z\}$ and $C = \{p, s, t, v, x, y\}$

i) $A - B = \{p, u, x\}$, $A' = \{r, s, t, w, z\}$, $B' = \{p, r, s, t, u, x\}$

$B' - A' = \{p, u, x\}$ and hence, $A - B = B' - A'$

ii) $B - A = \{z\}$ and so, $(A - B) \cup (B - A) = \{p, u, x, z\}$

Again, $A \cup B = \{p, q, u, v, x, y, z\}$ and $A \cap B = \{q, v, y\}$

So, $(A \cup B) - (A \cap B) = \{p, u, x, z\}$

Thus, $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

iii) $A - C = \{q, u\}$ and so, $(A - B) \cup (A - C) = \{p, q, u, x\}$

$B \cap C = \{v, y\}$ and so, $A - (B \cap C) = \{p, q, u, x\}$

Thus $A - (B \cap C) = (A - B) \cup (A - C)$.

Ans. to Q. No. 14 : i) $x \in B - A \Rightarrow x \in B$ and $x \notin A \Rightarrow x \in U$ and $x \notin A$

$\Rightarrow x \in A'$,

where x is an arbitrary element of $(B - A)$. Hence $B - A \subseteq A'$

ii) $x \in B - A' \Leftrightarrow x \in B$ and $x \notin A'$

$\Leftrightarrow x \in B$ and $x \in A \Leftrightarrow x \in B \cap A$

Hence $B - A' \subseteq B \cap A$ and $B \cap A \subseteq B - A'$

Thus, $B - A' = B \cap A$

iii) $A \subseteq A \cup B \Rightarrow A \subseteq \phi$, as $A \cup B = \phi$ ......(1)

Also $\phi \subseteq A$ ......(2)

From (1) & (2), we get $A = \phi$. Similarly, $B = \phi$.

1.16 FURTHER READINGS

1) *Discrete Mathematics* – Semyour Lipschutz & Marc Lipson.
2) *Discrete Mathematical Structures with Applications to Computer Scinece* – J. P. Tremblay & R. Manohar.

1.17 MODEL QUESTIONS

Q.1. Give examples of i) five null sets

ii) five finite sets

iii) five infinite sets

Q.2. Write down the following sets in rule method :

i) $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$

ii) $B = \left\{\frac{1}{1.2}, \frac{1}{1.3}, \frac{1}{3.4}, \frac{1}{4.5}, \ldots\right\}$

iii) C = {2, 5, 10, 17, 26, 37, 50}

Q.3. Write down the following sets in roster method

i) $A = \{x : x \in N, 2 < x < 10\}$

ii) $B = \{x : x \in N, 4+x < 15\}$

iii) $C = \{x : x \in Z, -5 \leq x \leq 5\}$

Q.4. If A = {1, 3}, B = {1, 3, 5, 9}, C = {2, 4, 6, 8} and D = {1, 3, 5, 7, 9} then fill up the dots by the symbol $\subseteq$ or $\not\subseteq$:

i) A ... B, ii) A ... C, iii) C ... D, iv) B ... D

Q.5. Write true or false :

i) $4 \in \{1, 2, \{3, 4\}, 5\}$, (ii) $\phi = \{\phi\}$

iii) A = {2, 3} is a proper subset of B = {x : (x–1)(x–2)(x–3) = 0}

iv) $A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$

Q.6. If $U = \{x : x \in N\}$, $A = \{x : x \in N, x \text{ is even}\}$, $B = \{x : x \in N, x < 10\}$ $C = \{x : x \in N, x \text{ is divisible by } 3\}$, then find

i) $A \cup B$, ii) $A \cap C$, iii) $B \cap C$, iv) A', (v) B', vi) C'.

Q.7. If $A \cup B = B$ and $B \cup C = C$, then show that $A \subseteq C$.

Q.8. If U = {–5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5},

A = {–5, –2, 1, 2, 4}

B = {–2, –3, 0, 2, 4, 5}

C = {1, 0, 2, 3, 4, 5},

then find i) $A \cup B$, ii) $A \cap C$, iii) $A \cap (B \cup C)$, iv) $B \cap C'$, v) $A' \cup (B \cap C')$, vi) $A - C'$, vii) $A - (B \cup C)'$, viii) $(B \cup C')$, ix) $A' \cup C'$, x) $(C' \cup B) - A$.

Q.9. Prove the following :

i) If A, B, C are three sets such that $A \subseteq B$, then $A \cup C \subseteq B \cup C$, $A \cap C \subseteq B \cap C$.

ii) $A \subseteq B$ if and only if $B' \subseteq A'$.

iii) $A \subseteq B$ if and only if $A \cap B = A$.

iv) If $A \cap B = \phi$, then $A \subseteq B'$.

Q.10. How many elements are there in P(A) if A has

i) 5 elements, ii) 2^n elements?

Q.11. Every resident in Guwahati can speak Assamese or English or both. If 80% can speak Assamese and 30% can speak both the language, what percent of residents can speak English?

Q.12. 76% of the students of a college drink tea and 63% drink coffee. Show that a minimum of 39% and a minimum of 63% drink both tea and coffee.

Q.13. In a survey of 100 students it is found that 40 read Readers' Digest, 32 read India Today, 26 read the Outlook, 10 read both Readers' Digest and India Today, 6 read India Today and the Outlook, 7 read Readers' Digest and the Outlook and 5 read all the three. How many read none of the magazines?

Q.14. In an examination 60% students passed in Mathematics, 50% passed in Physics, 40% passed in Computer Science, 20% passed in both Mathematics and Physics, 40% passed in both Physics and Computer Science, 30% passed in both Mathematics and Computer Science and 10% passed in all the three subjects. What percent failed in all the three subjects?

UNIT 2 : RELATIONS

UNIT STRUCTURE

2.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- know about the concepts of relation in a set
- learn about the types of relations in a set
- describe properties of relations in a set
- describe partitions set.

2.2 INTRODUCTION

In many problems concerning discrete objects, we find that there exists some kind of relationships among the objects. For example, there is a relationship between the employee and his salary, students and teachers, computer programs if they have some common data etc. 'Relation' has got tremendous application in every sphere of fields – social, economic, engineering, technological, etc. In computer science, the concept of relation is a major tool to understand it clearly. In this unit, we will introduce you to the concept of relation and some of its properties.

2.3 CARTESIAN PRODUCTS

Let A and B be two sets. The ***cartesian products*** of A and B, denoted by A x B, is the set of all ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.

Example: Let A = {a, b} and B = {a, c, d}.

Then A x B = {(a, a), (a, c), (a, d), (b, a), (b, c), (b, d)}

Similarly, if A = {1, 2, 3, 4) and B = {2, 7}

then, A x B = {(1,2), (1, 7), (2, 2), (2, 7), (3, 2), (3, 7), (4, 2), (4, 7)}

Also, B x A = {(2, 1), (2, 2,), (2, 3), (2, 4), (7, 1), (7, 2), (7, 3), (7, 4)}

NOTE

The sets A×B and B×A are not equal, unless A = B. It set A has *n* elements and B has *m* elements, then their product A×B has *nm* elements.

2.4 RELATIONS

We are familiar with human relations such as 'is brother of', 'is sister of', ' is son of' etc. We are also familiar with relations existing between arithmetical, geometrical or algebraic quantities such as '2 is less than 5', 'ΔABC is similar to ΔPQR', '(x+1) divides $(x^2–1)$', etc. These examples show that two quantities taken in a definite order gives us a relation.

Let us consider the set A = {1, 2, 3} and B = {4, 5, 6}

We have A x B = {(1, 4), (1, 5), (1, 6), (2, 4), (2, 6), (3, 6)}

Consider the subset R of A x B given by

R = {(1, 4), (1, 5), (1, 6), (2, 4), (2, 6), (3, 6)}

= {(x, y) : $x \in A$, $y \in B$ and x divides y}

The subset R of A x B establishes a relation from set A to set B.

We now generalise this concept of relation between two sets in the following subsection.

2.4.1 Relation in a Set

Let A and B be any two sets. A **binary relation** or, simply **a relation** from A to B is defined to be a subset of A x B. Generally, a relation is denoted by R. Thus, if R be a relation from A to B, then

$R \subseteq A \times B$

$R = \{(x, y) : x \in A, y \in B\}$

For $a \in A$, $b \in B$ if $(a, b) \in R$, we say that 'a is related to b under the relarion R' or, 'a is R-related to b', written as *aRb*.

If $(a, b) \notin R$, we say that 'a is not R-related to b' and we write as $a \not R b$.

Example : Let A = {x, y, z}, and B = {a, b, c}

and R = {(x, b), (x, c), (y, a), (z, c)}

Clearly $R \subset A \times B$ and hence R is a relation from A to B, where xRb, xRc, yRa, zRc.

As $(x, a) \notin R$, so $x \in a$. Similarly, $y \in b$, $z \in b$, etc.

2.4.2 Relation on a Set

Let A be a given set. If R be a relation from A to A, i.e. $R \subseteq A \times A$, then R is called a **(binary) relation** on A.

Example : Let A = { 2, 3, 4, 5, 6, 7, 8, 9 }.

Then R = {(2, 4), (2, 6), (2, 8), (3, 6), (3, 9), (4, 8)} is a relation on A. We can write $R = \{(x, y) : x, y \notin A \text{ and } x \text{ divides } y\}$

2.4.3 Domain and Range of a Relation

The Domain D, denoted by **Dom(R)**, of a relation R is defined as the set of all first elements of the ordered pairs which belong to R. i.e. $D = \{x \in A : (x, y) \in R, \text{ for } y \in B\}$

The Range E, denoted also by **Ran(R)**, of the relation R is defined as the set of all second elements of the ordered pairs which belong to R, i.e.

$E = \{y \in B : (x, y) \in R, \text{ for some } x \in A\}$

Obviously, $D \subseteq A$ and $E \subseteq B$.

Example: Let A = {1, 2, 3, 4} and B = {a, b, c}. Every subset of A x B is a relation from A to B. So, if R = {(2, a), (4, a), (4, c)} then domain of R is the set {2, 4} and the range of R is the set {a, c}.

Example : Let A = (2, 3, 4} and B = {3, 4, 5}. A relation R is defined as aRb if and only if a<b.

Then R = {(2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)}.

Dom(R) = {2, 3, 4} and Ran(R) = {3, 4, 5}.

Example: Let A = (2, 3, 5} and B = {3, 4, 5}. A relation R is defined as aRb if and only if a and b are both odd numbers.

Then, R = {(3, 3), (3, 5)}.

Dom(R) = {3} and Ran(R) = {3, 5}.

2.4.4 Total Number of Distinct Relations

Let the number of elements of the set A and B be *m* and *n* respectively. Then the number of elements of A x B is *mn*. Therefore, the number of elements of the power set of A x A is 2^{mn}. Thus, AxB has 2^{mn} different subsets. Now, every subset of A x B is a relation from A to B. Hence, the number of different relationA to B is 2^{mn}.

2.4.5 Some Operations on Relations

Let R and S be two relations from set A to set B.

Then $R \subseteq A \times B$, $S \subseteq A \times B$.

Clearly $R \cup S \subseteq A \times B$, $R \cap S \subseteq A \times B$, $R - S \subseteq A \times B$, $R' \subseteq A \times B$.

These are again relation from A to B, where

$R' = \{ (a, b) \in A \times B : (a, b) \notin R \}$

$R \cup S = \{ (a, b) : (a, b) \in R \text{ or } (a, b) \in S \}$

$R \cap S = \{ (a, b) : (a, b) \in R \text{ or } (a, b) \in S \}$

$R - S = \{ (a, b) : (a, b) \in R \text{ or } (a, b) \notin S \}$

Alternatively,

$a(R')b \Leftrightarrow a \not R b$

$a(R \cup S)b \Leftrightarrow aRb \text{ or } aSb$

$a(R \cap S)b \Leftrightarrow aRb \text{ and } aSb$

$a(R - S)b \Leftrightarrow aRb \text{ and } a \not S b$

Example : Let $A = \{ 1, 2, 3, 4 \}$, $B = \{ a,b,c \}$,

$R = \{ (1, a), (1, b), (2, c), (3, a), (3, b)\}$

and $S = \{ (1, b), (2, a), (3, a), (4, b)\}$

then $R' = \{(1, c), (2, a), (2, b), (3, c), (4, a), (4, b), (4, c)\}$

$R \cup S = \{(1, a), (1, b), (2, a), (2, c), (3, a), (3, b), (4, b)\}$

$R \cap S = \{(1, b), (3, a)\}$

$R - S = \{(1, a), (2, c), (3, b)\}$

CHECK YOUR PROGRESS

Q.1. Let P = {North, South, West} and Q = {House, Garden}.
Find (i) P x Q (ii) Q x P (iii) P x P

Q.2. If a set S has n elements, how many relations are there from S to S?

Q.3. Given A = {2, 3}, B = {3, 4, 5} be two sets and
R = {(2, 3), (2, 4), (3, 5)} be a relation. Find
(i) Domain(R) (ii) Range(R) (iii) Dom(R^{-1}) (iv) Ran(R^{-1})

2.4.6 Types of Relations in a Set

Here, we shall consider some special types of relations in a set.

Inverse Relation : Let R be a relation from the set A to the set B. The inverse of R, denoted by $\mathbf{R^{-1}}$ is the relation from the set B to A and is defined by

$R^{-1} = \{(b, a) : (a, b) \in R\}$

NOTE

Every relation has an inverse relation. If R be a relation from the set A to B, then $(R^{-1})^{-1} = R$.

That is, the inverse relation R^{-1} consists of those ordered pairs which when reversed, belong to R. Thus, every relation R from the set A to the set B has an inverse relation R^{-1} from B to A.

i.e. $xRy \Rightarrow y R^{-1}x$

Example: Let A = {a, b, c}, B = {x, y} and

R = {(a, x), (a, y), (b, x), (c ,y)} be a relation from A to B.

The inverse of R is R^{-1} = {(x, a), (y, a), (x, b), (y, c)}

Example: Let A = {2, 3, 4} and R = {(x, y) : |x – y| = 1} be a relation in A. That is, R = {(2, 3), (3, 2), (3, 4), (4, 3)}.

The inverse relation is R^{-1} = {(3, 2), (2, 3), (4, 3), (3, 4)}

Theorem 1: If R be a relation from set A to B, then the domain of R is the range of R^{-1} and the range of R is the domain of R^{-1}.

Proof: Let $y \in$ domain of R^{-1}. Then $y \in B$ and there exists $x \in A$ such that $(y, x) \in R^{-1}$.

Now $(y, x) \in R^{-1} \Rightarrow (x, y) \in R$

$\Rightarrow y \in$ Range of R

Thus, domain of $R^{-1} \Rightarrow y \in$ range of R.

Therefore, domain of $R^{-1} \subseteq$ range of R.

Similarly, we can prove that range of R $\subseteq$ domain of R^{-1}.

Therefore, domain of R^{-1} = range of R.

In a similar manner it can be proved that the domain of R is equal to the range of R^{-1}.

Identity Relation : A relation R in a set A is said to be identity relation, denoted by I_A, if $I_A = \{(x, x) \mid x \in A\}$

Example: Let A = {1, 2, 3} then I_A = {(1, 1), (2, 2), (3, 3)} is an identity relation in A.

Universal Relation : A relation R in a set A is said to be universal relation if R is equal to A x A.

Example: Let A = {a, b, c} then

R = A x A = {(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)} is a universal relation in A.

Void Relation : A relation R in a set A is said to be void relation if R is a null set i.e. R = Φ.

Every subset of A x A is a relation in A. Since Φ is also a sub set of A x A, therefore, the null set Φ is also a relation in A.

Example: Let A = {2, 3, 5} and relation R be defined as aRb if and only if 'a divides b', then R = Φ ⊆ A x A is a void relation.

2.4.7 Properties of Relations in a Set

Reflexive relation : A relation R is called reflexive relation if (a, a) ∈ R, for all a ∈ A; that is, R is reflexive if every element in A is related to itself.

Thus, R is reflexive if aRa for all a ∈ A.

Example: Let A = {1, 2, 3}. Then

i) the relation R = {(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)} is a reflexive relation since for every a ∈ A, (a, a) ∈ R.

ii) the relation R = {(1, 1), (1, 2), (2, 3), (3, 3)} is not a reflexive relation since for 2 ∈ A, (2, 2) ∉ R.

Anti-reflexive or Irreflexive Relation : A relation R in a set A is said to be anti-reflexive or irreflexive if for every a ∈ A, (a, a) ∉ R. In other words, there is no a ∈ A such that aRa.

Example:

i) The relation R = {(1, 2), (1, 3), (2, 1), (2, 3)} in a set A = {1, 2, 3) is anti-reflexive since (a, a) ∉ R for every a ∈ A.

ii) The relation R = {(1, 1), (2, 3), (3, 4)} in the set A = {1, 2, 3, 4} is not anti-reflexive since (1, 1) ∈ R.

Symmetric Relation : A relation R in a set A is said to be symmetric relation if (a, b) ∈ R ⇒ (b, a) ∈ R.

Thus, R is symmetric if bRa holds whenever aRb holds.

Example:

i) R = {(1, 1), (1, 2), (1, 3), (2, 2), (2, 1), (3,1)} in a set A = {1, 2, 3} is a symmetric relation.

ii) R = {(x, y) ∈ R x R : |x – y| > 0} is a symmetric relation on R, the set of real numbers.

Remark: Since (a, b) ∈ R ⇒ (b, a) ∈ R^{-1}, thus R is symmetric relation if and only if R = R^{-1}

Anti-Symmetric Relation : A relation R in a set A is said to be anti-symmetric relation if (a, b) $\in$ R and (b, a) $\in$ R $\Rightarrow$ a = b

A relation R in a set A is not anti-symmetric if there exists elements a, b $\in$ A, a $\neq$ b such that (a, b) $\in$ R and (b, a) $\in$ R.

Example:

i) The relation R = {(1, 2), (2, 2), (2, 3)} in a set A = {1, 2, 3} is an anti-symmetric relation.

ii) The relation R = {(x, y) $\in$ R^2 | x $\leq$ y} is an anti-symmetric relation on R since x $\leq$ y and y $\leq$ x implies x = y.

Thus, (x, y) $\in$ R and (y, x) $\in$ R $\Rightarrow$ x = y.

iii) Let S be the set of straight lines in a plane. The relation R in S defined as 'x is perpendicular to y' is not anti-symmetric, since if straight line a is perpendicular to the straight line b, then b is perpendicular to a also. But a cannot be equal to b i.e. (a, b) $\in$ R and (b, a) $\in$ R but a $\neq$ b.

Transitive Relation : A relation R in a set A is said to be transitive relation if whenever (a, b) $\in$ R and (b, c) $\in$ R then (a, c) $\in$ R, i.e. aRb and bRc $\Rightarrow$ aRc. A relation R in a set is not transitive if there exists elements a, b, c $\in$ A, such that (a, b) $\in$ R, (b, c) $\in$ R but (a, c) $\in$ R.

Example:

i) Let S be the set of straight lines in a plane. The relation R in S defined by 'x is parallel to y' is transitive, because if a line x is parallel to the line y and if y is parallel to the line z, then x is parallel to z.

ii) The relation 'is less than' is transitive on the set of real numbers. If a < b and b < c then a < c for real numbers a, b, c.

iii) The relation R = {(1, 2), (2, 1), (2, 3), (3, 2)} in the set A = {1, 2, 3} is not transitive since (1, 2) $\in$ R and (2, 3) $\in$ R but (1, 3) $\notin$ R.

2.4.8 Equivalence Relation

A relation R in a set A is an equivalence relation in A if and only if–

i) R is reflexive i.e. for all a ∈ A, (a, a) ∈ R.

ii) R is symmetric i.e. (a, b) ∈ R ⇒ (b, a) ∈ R

ii) R is transitive i.e. (a, b) ∈ R and (b, c) ∈ R ⇒ (a, c) ∈ R

Example:

i) Let A be the set of all triangles in a plane. Let R be the relation in A defined as xRy if and only if x is congruent to y, for x ∈ A and y ∈ A. Here

a) xRx for every x ∈ A, since every triangle is congruent to itself. Thus, R is reflexive.

b) xRy ⇒ yRx, since if triangle x is congruent ot the triangle y, then y is congruent to x. Thus R is symmetric.

c) xRy and yRz ⇒ xRz, since the triangle x is congruent to y, and triangle y is congruent to z, so triangle x is congruent to z. Hence, R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation.

NOTE

The relation in example (ii) is called the relation of congruence which is defined as follows :

Let m > 1, G a positive integer. Then for a, b ∈ Z. '***a is congruent to b modulo m***', denoted by a ∗ b (mod m), if (a–b) is divisible by m. It can be proved that this is an equivalence relation [to prove it, replace 5 by m in (ii)]

ii) Let R be a relation in the set of integers Z defined by

R = {(x, y) : x ∈ Z, y ∈ Z, x – y is divisible by 5}

Here we have

a) For each x ∈ Z, x – x = 0 and 0 is divisible by 5, therefore, xRx, ∀ x ∈ Z.

b) Let xRy. Then xRy ⇒ x – y is divisible by 5
⇒ –(x – y) is is divisible by 5
⇒ y – x is divisible by 5
⇒ yRx

c) Let xRy and yRz.

Then, xRy and yRz

⇒ x – y is divisible by 5 and y – z is divisible by 5
⇒ (x – y) + (y – z) is divisible by 5
⇒ x – z is divisible by 5
⇒ xRz.

Since R is reflexive, symmetric and transitive, therefore, R is an equivalence relation.

Theorem 2: If R be an equivalence relation in a set A, then R^{-1} is also an equivalence relation in A.

Proof: Let R be an equivalence relation in a set A. Therefore, R is reflexive, symmetric and transitive.

Let a, b, c $\in$ A be any three elements. The relation R^{-1} is

i) reflexive: $(a, a) \in R^{-1}$ since for all $(a, a) \in R \Rightarrow (a, a) \in R^{-1}$

ii) symmetric: $(a, b) \in R^{-1} \Rightarrow (b, a) \in R^{-1}$

since, $(a, b) \in R^{-1} \Rightarrow (b, a) \in R$

$\Rightarrow (a, b) \in R$ as R is symmetric

$\Rightarrow (b, a) \in R^{-1}$

iii) transitive: $(a, b), (b, c) \in R^{-1} \Rightarrow (a, c) \in R^{-1}$

since $(a, b), (b, c) \in R^{-1} \Rightarrow (b, a), (c, b) \in R$

$\Rightarrow (c, b),(b, a) \in R$

$\Rightarrow (c, a) \in R$ as R is transitive

$\Rightarrow (a, c) \in R^{-1}$

Therefore, R^{-1} is reflexive, symmetric and transitive and hence R^{-1} is an equivalence relation in A.

2.4.9 Equivalence Classes or Equivalence Sets

Let A be a non-empty set and R be an equivalence relation in A. Also let a be an arbitrary element of A. Then the element $x \in A$ satisfying xRa constitutes a subset of A, called equivalence class of a, denoted by **[a]** or **cl(a)**.

Thus, $[a] = \{x : x \in A \text{ and } (x, a) \in R\}$

[a] is a non-empty subset of A since $a \in [a]$.

Example: The equivalence relation R = {(1, 2), (2, 1), (1, 1), (2, 2), (3, 3), (4, 4)} on S = {1, 2, 3, 4} has the following equivalence classes.

[1] = [2] = {1, 2}   [3] = {3}   [4] = {4}

Example: Let S be the set of all triangles in a plane and let R be an equivalence relation in S defined by ‘x is congruent to y’, x, y $\in$ S. When a $\in$ S, the equivalence class [a] is the set of all triangles of S congruent to the triangle a. Similarly, when b $\in$ S, then equivalence class [b] is the set of all triangles of S congruent to the triangle b.

Properties of Equivalence Classes :

Theorem 3 : Let A be a non-empty set and R be an equivalence relation in A and a and b be arbitrary elements of A. Then

i) $a \in [a]$

ii) If $b \in [a]$, then $[a] = [b]$

iii) $[a] = [b]$ if and only if $(a, b) \in R$

iv) Either $[a] = [b]$ or $[a] \cap [b] = \Phi$ i.e. two equivalence classes are either equal or disjoint.

Proof: i) R being an equivalence relation, it is reflexive, i.e. aRa and $[a] = \{x : x \in A \text{ and } xRa\}$

Hence, $aRa \Rightarrow a \in [a]$

ii) We have $b \in [a] \Rightarrow bRa$

Let x be any arbitrary element of [b].

Then $x \in [b] \Rightarrow xRb$. But R is transitive, therefore, xRb and $bRa \Rightarrow xRa \Rightarrow x \in [a]$.

Thus $x \in [b] \Rightarrow x \in [a]$. Hence $[b] \subseteq [a]$.

Again, let y be any arbitrary element of [a].

Then $y \in [a] \Rightarrow yRa$.

Since R is symmetric, therefore, $bRa \Rightarrow aRb$.

Now, R being transitive, yRa and $aRb \Rightarrow yRb \Rightarrow y \in [b]$

Thus $y \in [a] \Rightarrow y \in [b]$. Therefore $[a] \subseteq [b]$.

Hence $[a] = [b]$.

iii) We assume that $[a] = [b]$

Since R is reflexive, we have aRa.

Again $aRa \Rightarrow a \in [a] \Rightarrow a \in [b]$ since $[a] = [b] \Rightarrow aRb$

Hence $[a] = [b] \Rightarrow aRb$ i.e. $(a, b) \in R$.

Conversely, we assume that $(a, b) \in R$ i.e. aRb.

Let x be any arbitrary element of [a]. Then xRa. Since R is transitive we have xRa and $aRb \Rightarrow xRb \Rightarrow x \in [b]$

Thus, $x \in [a] \Rightarrow x \in [b]$, that is $[a] \subseteq [b]$.

Again, let y be any arbitrary element of [b].

Then $y \in [b] \Rightarrow yRb$.

R being symmetric $aRb \Rightarrow bRa$.

Again R being transitive, yRb and bRa $\Rightarrow$ yRa $\Rightarrow$ y $\in$ [a]

Therefore, y $\in$ [b] $\Rightarrow$ y $\in$ [a], that is [b] $\subseteq$ [a].

Hence [a] = [b].

iv) If [a] $\cap$ [b] = Φ then there is noting to prove.

So we assume that [a] $\cap$ [b] $\neq \Phi$.

Therefore, there exists an element x $\in$ A such that

x $\in$ [a] $\cap$ [b].

Now x $\in$ [a] $\cap$ [b] $\Rightarrow$ x $\in$ [a] and x $\in$ [b]

$\Rightarrow$ xRa and xRb

$\Rightarrow$ aRx and xRb, as R is symmetric

$\Rightarrow$ aRb, as R is transitive

$\Rightarrow$ [a] = [b] [by (iii)]

Therefore, [a] $\cap$ [b] $\neq \Phi \Rightarrow$ [a] = [b].

2.4.10 Partitions

Let S be a non-empty set. Then a partition of S is a collection of non-empty disjoint subsets of S whose union is S. For example, let $A_1, A_2 \ldots .A_n$ be non-empty subsets of S.

Then the set P = { $A_1, A_2, \ldots, A_n$ } is said to be a partition of S, if i) $A_1 \cup A_2 \cup \ldots \cup A_n = S$

ii) Either $A_i = A_j$ or $A_i \cap A_j = \Phi$ for $1 \leq i \leq j \leq n$

Example: Let us consider the set S = {1, 2, 3, …. 9, 10} and its subsets B_1 = {1, 3}, B_2 = {7, 8, 10}, B_3 = {2, 5, 6} and B_4 = {4, 9}.

The set P = {B_1, B_2, B_3, B_4} is a partition of S since

i) $B_1 \cup B_2 \cup B_3 \cup B_4 = S$ and

ii) For any two sets B_i, B_j we have $B_i \cap B_j = \Phi$, $i \neq j$.

Example: Let Z be the set of all integers. We know that xy(mod 5) is an equivalence relation in Z. We consider the set of five equivalence classes [0], [1], [2], [3], [4] where

[0] = {…, -10, -5, 0, 5, 10, …}

[1] = {…, -9, -4, 1, 6, 11, …}

[2] = {…, -8, -3, 2, 7, 12, …}

[3] = {…, -7, -2, 3, 8, 13, …}

[4] = {…, -6, -1, 4, 9, 14, …}

Obviously,

i) The sets are non-empty.

ii) The sets [0], [1], [2], [3], [4] are pair-wise disjoints.

iii) Z = [0] $\cup$ [1] $\cup$ [2] $\cup$ [3] $\cup$ [4]

Hence {[0], [1], [2], [3], [4]} is a partition of Z.

2.4.11 Relation Induced by a Partition of a Set

Corresponding to any partition of a set A it is possible to define a relation R in A by the requirement that xRy if and only if x and y belong to the same subset of A belonging to the partition. The relation R is then said to be induced by the partition.

Example: Let A = {3, 6, 9, …, 24}, B = {1, 4, 7, …, 25}, C = {2, 5, 8, …, 23} be the subsets of the set S = {1, 2, 3, 4, …., 25}.

Obviously, A $\cup$ B $\cup$ C = S and A $\cap$ B = A $\cap$ C = B $\cap$ C = Φ so that {A, B, C} is a partition of S. If R be the relation induced by this partition then we have xRy if and only if x and y belong to the same subsets of A, B, C.

Theorem 4 : (*Fundamental theorem on equivalence relation*) An equivalence relation R in a non-empty set A forms a partition of A and conversely, a partition of A defines an equivalence relation in A.

Proof: Let R be an equivalence relation defiend in a non-empty set A. Let P be the set of equivalence classes of A with respect to the relation R. Thus P = {[a] : a $\in$ A} in which [a] = {x : x $\in$ A and xRa}.

Since R is an equivalence relation, for all a $\in$ A, we have aRa. Thus, a $\in$ [a] and [a] $\neq \Phi$.

Now every element a $\in$ A is an element of [a]. Hence we have A = [a], where a $\in$ A. Again equivalence classes are either identical or disjoint. Hence P is a partition of A which implies that an equivalence relation decomposes the set into equivalence classes, any two of which are either identical or disjoint.

Conversely, let P = {A_a, A_b, A_c, …} be a partition of the set A and let x, y $\in$ A. We define a relation R in A by xRy if and only if there is an A_i in P such that x, y $\in A_i$.

Let x be any arbitrary element of A. Then there exists $A_i \in P$ such that x $\in A_i$ i.e. xRx. Thus for all x $\in$ A, xRx i.e. R is reflexive.

Again, if xRy, then there exists $A_i \in P$

such that $x \in A_i$ and $y \in A_i$

But $x \in A_i$ and $y \in A_i \Rightarrow y \in A_i$ and $x \in A_i \Rightarrow yRx$

Therefore, R is symmetric.

Again let xRy and yRz. By definition of R there exists subsets A_j and A_k such that x, y $\in A_j$ and y, z $\in A_k$.(A_j and A_k are not necessarily distinct).

Now $y \in A_j$ and also $y \in A_k$, therefore $A_j \cap A_k \neq \Phi$.

But A_j and A_k belong to the partition P of A. So, $A_j \cap A_k \neq \Phi$ implies $A_j = A_k$. Now $A_j = A_k$ implies x, z $\in A_j$ and hence we have xRz.

Therefore, xRy and yRz $\Rightarrow$ xRz i.e. R is transitive.

Thus, R being reflexive, symmetric and transitive is an equivalence relation.

2.4.12 Quotient Set

Let A be a non-empty set and let R be an equivalence relation in A. The set of all mutually disjoint equivalence classes in which A is partitioned relative to the equivalence relation R, is said to be the ***quotient set*** of A for the equivalence relation R, and is denoted by A/R or $\overline{A}$.

Example: The quotient set of Z for the equivalence relation congruence modulo 5 is the set Z/R = {[0], [1], [2], [3], [4]}.

2.4.13 Partial Order Relation

Let A be a non-empty set. A relation R on A is said to be a partial order relation if R is reflexive, anti-symmetric and transitive. That is,

1. Reflexivity: aRa for all $a \in A$
2. Anti-symmetry : aRb and bRa $\Rightarrow$ a = b
3. Transitivity : aRb and bRc $\Rightarrow$ aRc

A non-empty set A together with a relation of partialy ordered R is called a partial order set or poset and is denoted by (S, R).

Example: The greater than or equal ($\geq$) relation is a partial order relation on Z, the set of integers.

Reflexive: since $a \geq a$ for every integer a

Anti-symmetric: since $a \geq b$ and $b \geq a$ imply a = b

Transitive: since $a \geq b$ and $b \geq c$ imply $a \geq c$.

Hence, $\geq$ being reflexive, anti-symmetric and transitive is a partial ordering on Z and (Z, $\geq$) is a poset.

CHECK YOUR PROGRESS

Q.4. Given S={1, 2, 3, 4}. Consider the following relation in S :
S = {(1, 1), (2, 2), (2, 3), (3, 2), (4, 2), (4, 4)}
Is R (i) reflexive, (ii) symmetric (iii) transitive or (iv) anti-symmetric?

Q.5. Determine which of the following are equivalence relations and / or partial ordering relations for the given sets?
i) G = {lines in the plane : xRy $\Rightarrow$ x is parallel to y}
ii) H = {the set of real numbers : xRy $\Rightarrow$ $|x - y| \leq 7$}

2.5 LET US SUM UP

- The ***cartesian products*** of A and B, denoted by A x B, is the set of all ordered pairs of the form (a, b) where $a \in A$ and $b \in B$.
- Let A and B be two sets. *A relation from set A to B is a subset of the cartesian product A x B*.
- The Domain D, denoted by **Dom(R)**, of a relation R is defined as the set of all first elements of the ordered pairs which belong to R.

- The Range E, denoted also by **Ran(R)**, of the relation R is defined as the set of all second elements of the ordered pairs which belong to R.
- Every relation R from the set A to the set B has an inverse relation R^{-1} from B to A. i.e. $xRy \Rightarrow y\, R^{-1}x$
- A relation R in a set A is said to be
  - i) reflexive if aRa $\forall$ a $\in$ A
  - ii) symmetric if aRb $\Rightarrow$ bRa for a,b $\in$ A
  - iii) transitive if aRb, bRc $\Rightarrow$ aRc for a, b, c $\in$ A
  - iv) anti-symmetric if aRb, bRa $\Rightarrow$ a = b for a, b $\in$ R
- A relation R in a set A is an equivalence relation in A if and only if – R is reflexive, Symmetric and Transitive.

2.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 : i) P x Q = {(North, House), (North, Garden), (South, House), (South, Garden), (West, House), (West, Garden)}

ii) Q x P = {(House, North), (House, South), (House, West), (Garden, North), (Garden, South), (Garden, West)}

iii) P x P = {(North, North), (North, South), (North, West), (South, North), (South, South), (South, West), (West, North), (West, South), (West, West)}

Ans. to Q. No. 2 : 2^n

Ans. to Q. No. 3 : i) {2,3} ii) {3,4,5} iii) {3,4,5} iv) (2,3}

Ans. to Q. No. 4 : i) No ii) No iii) No iv) No

Ans. to Q. No. 4 : i) It is an equivalence relation but not a partial ordering relation since R is not anti-symmetric.

ii) Not transitive, therefore, it is neither.

2.7 FURTHER READINGS

1. C. L. Liu, *Elements of Discrete Mathematics*, Tata McGraw-Hill Edition.

2. Seymour Lipschutz, Marc Lars Lipson, *Discrete Mathematics*, Tata McGraw-Hill Edition.

2.8 MODEL QUESTIONS

Q.1. The relation R on the set {1, 2, 3, 4, 5} is defined by the rule that (x, y) $\in$ R if 3 divides x – y. Find

i) The element of R
ii) The elements of R^{-1}
iii) The domain of R
iv) The range of R
(v) The domain of R^{-1}
vi) The range of R^{-1}

Q.2. Give example of a relation which is

i) symmetric, reflexive but not transitive
ii) symmetric and transitive but not reflexive
iii) reflexive, transitive but not symmetric
iv) neither symmetric nor anti-symmetric

Q.3. Show that the relation R is on a set S is symmetric if and only if the converse relation R^{-1} is symmetric.

Q.4. Show that the relation R ‘less than or equal to’ on the set of integers is a partial order relation.

Q.5. Let S be the set of non-zero integers and let R be the relation on SxS, defined by (a,b) $\in$ R(c,d) $\Leftrightarrow$ ad = bc

i) Show that R is an equivalence relation
ii) Find [(1,2)]

Q.6. In the set N of positive integers, we say that “a divides b”, in symbol a/b, if b = ak for some k $\in$ N. Show that this relation is a partial order in N.

UNIT 3 : FUNCTIONS

UNIT STRUCTURE

3.1 LEARNING OBJECTIVES

After going through this unit, you will be able to :

- learn about the concepts of functions
- differentiate between relation and function
- learn the types of funtions
- describe composite functions and its properties
- describe binary operation and its different types.

3.2 INTRODUCTION

One of the most important concepts in mathematics is that of a *'function'*. It is a special kind of relation between two non-empty sets. Informally, a function is an "operation" which takes an input value and gives an output value. There are other terms such as 'map' or 'mapping' used to denote a function. From the computer science perspective, functions play a vital role. In this unit, we will introduce you to the concepts of functions. Moreover, we shall discuss about the types of function and composite functions and its properties.

3.3 FUNCTIONS

A function or mapping *f* from a set A to set B denoted by $f : A \to B$ is a rule that assigns to each element x in A exactly one element y in B.

The function is read as '*f* is a function from A to B'.

A function can also be denoted by $f : A \to B$ *or* $A \xrightarrow{f} B$ or

A function can be represented diagramatically as

> For a function $f : A \to B$ each element in the set A must be associated to one and only one element in set B. But, there may be some elements in the set B which are not associated to any element in the set A.

Example: Let A = {1, 2, 3, 4} and B = {1, 4, 9, 16} be two sets and *f* assigns to each element *x* of A a unique element x^2 in B. Then *f* is a function from A to B. Thus, we may write

$f(1) = 1$, $f(2) = 4$, $f(3) = 9$ and $f(4) = 16$.

The function *f* is defined as $f(x) = x^2 \ \forall \ x \in A$.

Example: Let A = {2, 3, 4} and B = {4, 9} and if *f* assigns to each element of A its square values then *f* is not a function from A to B, since no number of B is assigned to the element $4 \in A$.

3.4 RANGE AND DOMAIN OF A FUNCTION

If *f* is a function from a set A to the set B, then the set A is called the ***domain*** of the function *f* and the set B is called the ***co-domain*** of the function *f*. The element y $\in$ B which the function *f* associates to an element x $\in$ A is denoted by *f*(x) and y = *f*(x) is called the *image* of x under *f* or *f-image* or the *value of the function f for x*. The element x is called the *pre-image* of y or *f*(x).

The set *f*(A) = {*f* (x) | x $\in$ A} consisting of all images of the elements in A under the function *f* is called the *range* of *f*. Clearly, *f*(A) $\subseteq$ B.

Example: Let A = {–1, 1, 3, 4} and B = {1, 2, 9, 16} and be a function defined by *f*(x) = x^2 for all x in A.

Domain of *f* = {–1, 1, 3, 4}

Co-domain of *f* = {1, 2, 9, 16}

Range of *f* = {1, 9, 16}

NOTE

For a function *f :* A→B each element in A has a unique image and each element in B need not appear as the image of an element in A. There may be more than one element of A which have the same image in B.

3.5 FUNCTIONS AS SETS OF ORDERED PAIRS

A function *f* from a set A to a set B is a relation such that every a $\in$ A is related to exaclty one b $\in$ B. That is,

i) $\forall$ a $\in$ A, $\exists$ b $\in$ B such that (a, b) $\in$ *f*

ii) For a $\in$ A, and b_1, $b_2 \in$ B; (a, b_1) $\in$ *f* , (a, b_2) $\in$ *f* $\Rightarrow$ $b_1 = b_2$

Example: Let A = {a, b, c, d}, B = {p, q, r}.

Then *f* = {(a, p), (b, q), (c, p), (d, r)} is a function from A to B ,

where *f*(a) = p , *f*(b) = q , *f*(c) = p, *f*(d) = r

Example: Let A = {a, b, c, d}, B = {p, q, r}.

Then *f* = {(a, p), (b, q), (a, r), (d, r)} is not a function since p and r in B are assigned to the same element a $\in$ A.

Example: Let A = {a, b, c, d}, B = {p, q, r}.

Then *f* = {(a, p), (b, q), (c, p)} is not a function since no element in B is assigned to the element d $\in$ A.

> **NOTE**
> If A, B are subsets of the real numbers set R, then a function $f : A \to B$ is called a real valued function of real numbers. Generally, this function is simply denoted by f(x) for $x \in R$. Its domain is taken as the largest subset of R for which $f(x) \in R$.

3.6 DIFFERENCE BETWEEN RELATIONS AND FUNCTIONS

We know that every subset of A x B is a relation from A to B. Thus every function is a relation, but every relation is not a function. In a relation from A to B, an element of A may be related to more than one element in B. Also, there may be some elements of A which may not be related to any element in B. But in a function from A to B, each element in A must be associated to one and only one element in B.

Example : Let X = {1, 2, 3, 4} and Y = {a, b, c} be two sets.

Then *f* = {(1, a), (2, b), (3, c), (4, c)} is a function as well as a relation. But if we consider a subset S of X × Y as

S = {(1, a), (2, b), (1, c), (3, c), (4, b)}, then S is a relation from A to B. But, S is not a function since 1 in X is associated with two different elements a and c in Y.

CHECK YOUR PROGRESS

Q.1. A function $f : A \to R$ is defined as $f(x) = \begin{cases} 3x-4 & x>0 \\ -3x+2 & x \le 0 \end{cases}$

Determine *f*(0), *f*(2/3), *f*(–2).

Q.2. Let A = {–2, –1, 0, 1, 2}. A function $f : A \to R$ is defined as $f(x) = x^2 + 1$. Find the range of *f*.

Q.3. Determine whether or not the followins are functions from A to B where A = {1, 2, 3, 4, 5} and B = {a, b, c, d, e}

i) f_1 = {(1, a), (2, b), (3, b), (5, e)}

ii) f_2 = {(1, e), (5, d), (3, a), (2, b), (1, d), (4, a)}

iii) f_3 = {(5, a), (1, e), (4, b), (3, c), (2, d)}

Q.4. Find the domain D of each of the follwoing real-valued functions of a real variable :

i) $f(x) = \dfrac{1}{x-2}$ ii) $f(x) = \sqrt{25 - x^2}$

iii) $f(x) = x^2 - 3x - 4$ iv) $f(x) = x^2$ where $0 \le x \le 2$

3.7 TRANSFORMATION OR OPERATORS

If the domain and co-domain of a function *f* are both the same set i.e., $f : A \to A$, then *f* is called an operator or transformation on A.

Example : $f : N \to N$, $f(n) = 2n+1 \ \forall \ n \in N$ is an operator on N.

3.8 EQUALITY OF TWO FUNCTIONS

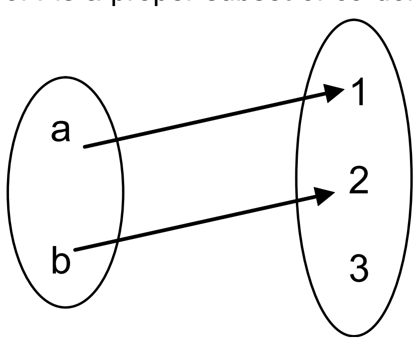
Two functions *f* and *g* defined on the same domain A are said to be equal if and only if $f(x) = g(x)$ for all x in A and we write $f = g$.

Example : Let $f : N \to N$, $g : N \to N$, $h : N \to N$ be three functions defined by f(n) = 2n, g(n) = 2m, h(n) = 2n + 1. then f = g, $f \neq h$.

3.9 TYPES OF FUNCTIONS OR MAPPINGS

3.9.1 Into Function

A function $f : A \to B$ is called *into function* if there exists at least one element in B which is not the image of any element in A i.e., the range of *f* is a proper subset of co-domain of *f*.



Example: Let Z be the set of integers and $f : Z \to Z$ be defined by $f(x) = 2x \ \forall \ x \in Z$. Then *f* is an into function because $f(Z) \subset Z$.

3.9.2 Onto Function

A function $f : A \to B$ is called onto or surjective if every element in B is the image of at least one element of A, i.e., the range of *f* is equal to the co-domain of *f*.

To prove that f : A→B is onto, we show that for y ∈ B, ∃ x ∈ A such that y = f(x). Then y ∈ B ⇒ y ∈ f(A). Having chosen y arbitrarily, every element of B is an element of f(A) and hence B ⊆ f(A). But f(A) ⊆ B. Therefore, B = f(A) and the function f is onto.

Example: Let f : R→R be a function defined by f(x) = x^2, ∀ x ∈ *R.* f is not an onto function because there is no real number whose square root is negative. Hence the range of f cannot be equal to *R.*

Example: Let f : Z→Z be defined by f(x) = x + 1, x ∈ Z. Then every element y in co-domain set *Z* has a pre-image y–1 in domain set *Z*. Thus, *f(Z)* = *Z* and f is an onto function.

3.9.3 One-One Function

A function f : A→B is called a bijective function if f is both one-one and onto.

A function f : A→B is said to be one-one or one to one if different elements in A have different images in B. i.e., $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, where $x_1, x_2 \in A$. Equivalently, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

An one-one function is also known as *injective function*.

If we are to prove that a function f is one-one, we have to show that if $f(x_1) = f(x_2)$ then $x_1 = x_2$, where x_1 and x_2 are arbitrary elements of domain of function f. We can also prove it by showing that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

Example: Let $f : R \to R$ be defined by $f(x) = 2x + 1, x \in R$.

Then for $x_1, x_2 \in R$ and $x_1 \neq x_2$ we have $f(x_1) \neq f(x_2)$.

So f is an one-one function.

3.9.4 Many-One Function

A function $f : A \to B$ is said to be many-one if two or more elements in A have same images in B. i.e. $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ where $x_1, x_2 \in A$.

Example: Let $f : R \to R$ be defined by $f(x) = x^2, x \in R$.

Then $f(1) = 1^2 = 1$ and

$f(-1) = (-1)^2 = 1$

Thus $f(-1) = f(1)$, but $-1 \neq 1$ and hence f is many-to-one.

3.9.5 Constant Function

A function $f : A \to B$ is said to be a constant function if a single element b B is assigned to each element in A. i.e. $f : A \to B$ is a constant function if the range of f consists of only one element, $f(x) = b \ \forall x \in A$.

Example: $f : R \to R$ be defined by $f(x) = 3, x \in R$ is a constant function since 3 is assigned to each element in A.

3.9.6 Identity Function

A function $f : A \to A$ defined by $f(x) = x$ for every x in A is called an identity function of A. It is generally denoted by I_A, i.e.,

$I_A : A \to A, I_A(x) = x \ \forall x \in A$.

Example: Let A = {a, b, c, d}. Then f = {(a, a), (b, b), (c, c), (d, d)} is an identity function in A.

3.9.7 Inverse Function

Inverse Image of an Element : Let f be a function from A to B and let $b \in B$. Then inverse image of element b under f is denoted by $f^{-1}(b)$

NOTE

Only one-one and onto function possesses inverse function because
(i) if the function $f : A \to B$ is not onto, then there will be some elements in B which will have no pre-images in A.
(ii) if the function $f : A \to B$ is not one-one then the elements in B will be assignd more than one element in A.

(read as 'f inverse b') and consists of those elements in A which have b as their f-image.

Thus if $f : A \to B$, then $f^{-1}(b) = \{x \in A \mid f(x) = b\}$

Clearly, $f^{-1}(b) \subseteq A$.

Example: Let $f : R \to R$ be a function defined by $f(x) = x^2$, $x \in R$. Then $f^{-1}(25) = \{-5, 5\}$.

Inverse Function : Let $f : A \to B$ be a one-one and onto function. Then the function $f^{-1} : B \to A$ which assigns to each element $b \in B$ the element $a \in A$ such that $f(a) = b$, is called the inverse function of $f : A \to B$. Clearly, $f^{-1}(b) = a \Leftrightarrow f(a) = b$.

Example: Let A = {1, 2, 3, 4} and B = {a, b, c, d} and let be given by $f = \{(1, a), (2, a), (3, d), (4, c)\}$. Then $f^{-1} : B \to A$ is not a function since $f^{1}(a) = \{1, 2\}$

Example: Let the function $f : R \to R$ be defined by $f(x) = x^2$.

$$\begin{aligned} \text{Then } f^{-1}(4) &= \{x \in R : f(x) = 4\} \\ &= \{x \in R : x^2 = 4\} \\ &= \{x \in R : x = \pm 2\} \\ &= \{-2, 2\} \\ \text{and } f^{-1}(-4) &= \{x \in R : f(x) = -4\} \\ &= \{x \in R : x^2 = -4\} \\ &= \{x \in R : x = \pm 2\sqrt{-1}\} = \phi \end{aligned}$$

since $\pm 2\sqrt{-1}$ is an imaginary number.

Theorem: If $f : A \to B$ is one-one and onto function, then $f^{-1} : B \to A$ is also one-one and onto.

Proof: First we shall show that f^{-1} is one-one.

Let y_1 and y_2 be any two elements in B such that $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$ where $x_1, x_2 \in A$. Then by definition of f^{-1}, $f(x_1) = y_1$ and $f(x_2) = y_2$

$$\begin{aligned} \text{Now } f^{-1}(y_1) = f^{-1}(y_2) &\Rightarrow x_1 = x_2 \\ &\Rightarrow f(x_1) = f(x_2) \\ &\Rightarrow y_1 = y_2 \end{aligned}$$

The function f^{-1} is one-one.

Now we show that f^{-1} is onto.

Let x be any arbitrary element of A. Since f is function from A to B, therefore there exists an element y $\in$ B such that y = *f*(x) or x = f^{-1}(y). Thus x is the f^{-1} image of the element y $\in$ B. Hence, the function f^{1} is also onto.

Theorem : If *f* : A$\rightarrow$B is one-one and onto, then the inverse function of f is unique.

Proof: Let *f* : A$\rightarrow$B is one-one and onto. Let *g* : B$\rightarrow$A and *h* : B$\rightarrow$A be two inverse funtions of *f*. To prove that *f* = *g*.

Let b be any arbitrary element in B. Let *g*(b) = a and *h*(b) = c. Since *g* is the inverse function of *f*, therefore, *g*(b) = a $\Rightarrow$ *f*(a) = b. Also since h is the inverse function of *f*, therefore, *h*(b) = c $\Rightarrow$ *f*(c) = b. But *f* is one-one. Therefore, *f*(a) = b and *f*(c) = b $\Rightarrow$ a = c $\Rightarrow$ *g*(b) = *h*(b). Hence *g* = *h* the inverse function of *f*, therefore, *h*(b) = c $\Rightarrow$ *f*(c) = b. But *f* is one-one. Therefore, *f*(a) = b and *f*(c) = b $\Rightarrow$ a = c $\Rightarrow$ *g*(b) = *h*(b). Hence *g* = *h*.

CHECK YOUR PROGRESS

Q.5. Show that the function *f* : R$\rightarrow$R defined as *f*(x) = cosx, x $\in$ R is neither one-one nor onto.

Q.6. Given that A = R – {3} and B = R – {1}, where R is the set of real numbers and *f* : A$\rightarrow$B is defined by $\frac{x-2}{x-3}$, x $\in$ A. Prove that *f* is one-one and onto.

Q.7. If the function *f* : R$\rightarrow$R is defined by *f*(x) = 2x – 3. Find f^{-1}.

3.10 COMPOSITION OF FUNCTIONS

Let *f* : A$\rightarrow$B and *g* : B$\rightarrow$C. Then the composition of *f* and *g* denoted by (*g* $\circ$ *f*) or *gf* is a function from A to C given by

(*g* $\circ$ *f*) : A$\rightarrow$C such that (*g* $\circ$ *f*)(x) = *g*[*f*(x)] $\forall$ x $\in$ A

Example: Let $f: R \to R$ and $g: R \to R$ be two functions given by

$f(x) = 2x + 1$ and $g(x) = x^2$

Now $(g \circ f): R \to R$ is given by

$(g \circ f)(x) = g[f(x)] = g(2x + 1) = (2x + 1)^2 = 4x^2 + 4x + 1$

and $(f \circ g)(x) = f[g(x)] = f(x^2) = 2x^2 + 1$

In general, $(g \circ f) \neq (f \circ g)$

Theorem: If $f: A \to B$, $g: B \to C$ and $h: C \to D$,

then $h \circ (g \circ f) = (h \circ g) \circ f$

Proof: Since $f: A \to B$ and $g: B \to C$ and $h: C \to D$,

so $(g \circ f): A \to C$ and $(h \circ g): B \to D$.

Also $h \circ (g \circ f): A \to D$ and $(h \circ g) \circ f: A \to D$

Let $x \in A$, $y \in B$, $z \in C$ such that $f(x) = y$ and $g(y) = z$.

Then $[(h \circ g) \circ f](x) = (h \circ g)[f(x)] = (h \circ g)(y)$

$= h[g(y)] = h(z)$ ......(i)

Also, $[(h \circ g) \circ f](x) = [h \circ (g \circ f)(x)] = h[(g \circ f)(x)$

$= h[g(f(x))] = h[g(y)] = h(z)$ ......(ii)

From (i) and (ii) we get

$[h \circ (g \circ f)](x) = [h \circ (g \circ f)](x)$ for all x in A.

Hence, $h \circ (g \circ f) = (h \circ g) \circ f$

Theorem: If $f: A \to B$ and $g: B \to C$ be one-one and onto functions, then $(g \circ f)$ is also one-one and onto and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof: Let x_1 and x_2 be any two elements in A.

Then $(g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow g[f(x_1)] = g[f(x_2)]$

$\Rightarrow f(x_1) = f(x_2)$

[since $g: B \to C$ is one-one and $f(x_1), f(x_2) \in B$]

$\Rightarrow x_1 = x_2$ [f is one-one]

Therefore, $(g \circ f)$ is one-one.

Let z be arbitrary element of C. Since is onto, $\exists y \in B$ such that $g(y) = z$. Again since is onto, $\exists x \in A$ such that $f(x) = y$. Thus for every $z \in Z$, $\exists x \in A$ such that . Hence, $z = g(y) = g[f(x)] = (g \circ f)(x)$. Hence, $(g \circ f)$ is onto. As is both one-one and onto, so $(g \circ f)^{-1}$ exists.

By definition of composite function $(g \circ f): A \to C$. So $(g \circ f)^{-1}: C \to A$. Also $g^{-1}: C \to A$ and $f^{-1}: B \to A$. The domain of $(g \circ f)^{-1}$ = the domain of $f^{-1} \circ g^{-1}$.

Now $(g \circ f)^{-1}(z) = x \Leftrightarrow (g \circ f)(x) = z$

$\Leftrightarrow g(f(x)) = z$

$\Leftrightarrow g(y) = z$

$\Leftrightarrow y = g^{-1}(z)$

$\Leftrightarrow f^{-1}(y) = f^{-1}(g^{-1}(z)) = (f^{-1} \circ g^{-1})(z)$

$\Leftrightarrow x = (f^{-1} \circ g^{-1})(z)$

Thus, $(g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)\ \forall\ z \in C$ and so $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

3.11 BINARY OPERATIONS

A binary operation '∗' on a non empty set A is a function which associates with each ordered pair (a, b) of elements of A, a uniquely defined element c ∈ A. Thus "∗" is a function of the product set A × A to A.

Closure Operations : A set A is said to be closed with respective to the operation ∗ if for all a, b ∈ A, a ∗ b ∈ A.

Commutative Operations : A binary operation on a set A is called commutative if a ∗ b = b ∗ a, for all a, b ∈ A.

Associative Operations : A binary operation ∗ on a set A is called associative if (a ∗ b) ∗ c = a ∗ (b ∗ c), for all a, b, c ∈ A.

Distributive Operations : Let A be a set on which two binary operations • and ∗ are defined. The operations is said to be left distributive with respective to • if a ∗ (b ∗ c) = (a ∗ b) • (a ∗ c), for all a, b, c ∈ A and is said to be the right distributive with respective to o if (b • c) a = (b ∗ a) • (c ∗ a), for all a, b, c ∈ A.

Identity Element : Let ∗ : A × A → A be a binary operation on A. An element e ∈ A is called an identity element for the operation ∗ if e ∗ a = a ∗ e = a, for all a ∈ A.

Inverse Element : An element a ∈ A is said to have an inverse element with respect to a binary operation ∗ with identity e if there exists b ∈ A such that a ∗ b = e = b ∗ a. Then b is called an inverse of a and is denoted by a^{-1}.

Cancellation laws : The cancellation laws, under the binary operation ∗ on a set A are

i) a ∗ b = a ∗ c ⇔ b = c (left cancellation law)

ii) b ∗ a = c ∗ a ⇔ b = c (right cancellation law)

CHECK YOUR PROGRESS

Q.8. Let $f : R \to R$ be given by $f(x) = x^2$ and $g : R \to R$ be given by $g(x) = x + 3$. Calculate $(f \circ g)(2)$ and $(g \circ f)(2)$.

Q.9. Let the functions $f : R \to R$ and $g : R \to R$ be defined by $f(x) = 2x$ and $g(x) = x^2 + 2$, $\forall x \in R$. Find the formulae defining the functions $(f \circ g)$ and $(g \circ g)$. Obtain $(f \circ g)(2)$ and $(g \circ g)(1)$.

3.12 LET US SUM UP

- A function or mapping f from a set A to set B denoted by $f : A \to B$ is a rule that assigns to each element x in A exactly one element y in B.
- If f is a function from a set A to the set B, then the set A is called the ***domain*** of the function f and the set B is called the ***co-domain*** of the function f.
- A function f from a set A to a set B is a relation such that every $a \in A$ is related to exaclty one $b \in B$.
- A function $f : A \to B$ is called *into function* if there exists at least one element in B which is not the image of any element in A.
- A function $f : A \to B$ is called onto or surjective if every element in B is the image of at least one element of A.
- A function $f : A \to B$ is said to be one-one or one to one if different elements in A have different images in B. i.e. $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ where $x_1, x_2 \in A$.
- A function $f : A \to B$ is said to be many-one if two or more elements in A have same images in B. i.e. $f(x_1) = f(x_2) \nRightarrow x_1 = x_2$ where $x_1, x_2 \in A$.
- A function $f : A \to A$ defined by $f(x) = x$ for every x in A is called an identity function of A.

3.13 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 : 2, –2, 8

Ans. to Q. No. 2 : {5, 2, 1}

Ans. to Q. No. 3 : i) No ii) No iii) Yes

Ans. to Q. No. 4 : i) D = R\{2} ii) D = $\{x \in R : -5 \le x \le 5\}$

iii) D = R iv) D = $\{x \in R : 0 \le x \le 2\}$

Ans. to Q. No. 5 : $f^{-1}(x) = \frac{x+3}{2}$

Ans. to Q. No. 6 : left as an exercise

Ans. to Q. No. 7 : left as an exercise

Ans. to Q. No. 8 : 25, 7.

Ans. to Q. No. 9 : $(f \circ g)(x) = 2x^2 + 4$, $(g \circ g)(x) = x^4 + 4x^2 + 6 \ \forall x \in R$.

$(f \circ g)(2) = 12$, $(g \circ g)(1) = 11$.

3.14 FURTHER READINGS

1. C. L. Liu, *Elements of Discrete Mathematics*, Tata McGraw-Hill Edition.
2. Seymour Lipschutz, Marc Lars Lipson, *Discrete Mathematics,* Tata McGraw-Hill Edition.

3.15 MODEL QUESTIONS

Q.1. Let A = {1, 2, 3} and B = {8, 9}

Find whether the following subsets of AxB are functions from A to B.

i) f = {(1, 8), (1, 9), (2, 8), (3, 9)}

ii) g = {(1,9), (2,9), (3,9)}

iii) h = {(1,8), (2, 9), (3, 9)}

Q.2. Let Q be the set of all rational numbers. Prove that the function $f : Q \to Q$ defined by $f(x) = 5x + 2$, $x \in Q$ is a bijective function. Find f^{-1}.

Q.3. Is the function $f : R - \{0\} \to \{-1, 1\}$ defined by $f(x) = \frac{x}{|x|}$ bijective?

Q.4. If $f : A \to B$ is invertible with inverse function $f^{-1} : B \to A$, then prove that $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B$.

Q.5. Show that the composition $a * b = ab^2$, $a, b \in R$ is not associative.

UNIT 4 : INTRODUCTION TO MATHEMATICAL LOGIC

UNIT STRUCTURE

4.1 LEARNING OBJECTIVES

After going through this unit, you will be able to :

- define statements and examples of statements
- define truth tables about different statements
- know about negation of statements
- know about conjuction, disjunction, conditional and bi-conditional of two statements
- learn about converse, opposite and contrapositive of statement.

4.2 INTRODUCTION

Mathematical logic or logic is the discipline that deals with the methods of reasoning. As we all know, the main asset that makes humans far superior to other species is the ability of reasoning. Logic provides rules and techniques for determining whether a given argument or mathematical proof or conclusion in a scientific theory is valid or not. Logic is concerned with studying arguments and conclusions. Logic is used in mathematics to prove theories and to draw conclusions from experiments in physical science in our every day life to solve many types of problem. Logic is used in computer science to verify the correctness of programs. The rules of logic or techniques of logic are called rules of inference because the main aim of logic is to draw conclusions and inferences from given set of hypothesis. In this unit, we will introduce you to the definition and examples of Statements, truth table of different statements. We will also discuss the logical connectives. Discussing besides the converse, opposite and contrapositive of statements.

4.3 DEFINITION OF STATEMENTS

We communicate our ideas or thoughts with the help of sentences in a particular language. The following types of sentences are normally used in our everyday communication.

Assertive sentence : A sentence that makes an assertion is called an assertive sentence or declarative sentence.

For example, “Mars supports life’’ is an assertive or a declarative sentence.

Imperative sentence : A sentence that expresses a request or a command is called an imperative sentence .

For example, “please bring me a cup of tea’’ is an imperative sentence.

Exclamatory sentence : A sentence that expresses some strong feelings is called an exclamatory sentence.

For example, “How big is the whale fish!” is an exclamatory sentence.

Interrogative sentence : A sentence that asks some questions is called an interrogative sentence.

For example, “What is your age ?’’ is an interrogative sentence.

STATEMENT : *A statement is an assertive (or declartive) sentence which is either true or false but not both. The truthfulness, denoted by T or the falsity, denoted by F of a statement is called the Truth value of the statement.*

4.3.1 Examples of Statements

ILLUSTRATION 1 : Consider the following sentences :

i) Washington D .C is not in America.

ii) Every quadrilateral is a rectangle.

iii) The earth is a planet.

iv) Three plus six is 9.

v) The sun is a star.

Each of the sentences (iii) , (iv) & (v) is a true declarative sentence and so each of them is a statement .

Each of the sentences (i) & (ii) is a false declarative sentence and so each of them is a statement .

ILLUSTRATION 2 : Consider the following sentences :

i) Do your home work.

ii) Give me a glass of water.

iii) How are you?

iv) Have you ever seen the Taj Mahal?

v) May god bless you!

vi) May you live long!

Sentences (i) & (ii) are imperative sentences, so they are not statements. Each of the sentences (iii) & (iv) is interrogative. So they cannot be statements. Similarly, (v) & (vi) are also not declarative sentences and hence not statements.

CHECK YOUR PROGRESS

Q.1. Find out which of the following sentences are statements and which are not – justify your answer.

i) Paris is in England.

ii) May God bless you !

iii) 6 has three prime factors.

iv) 18 is less than 16.

v) How far is Chennai from here?

vi) Every rhombus is a square.
vii) There are 35 days in a month.
viii) Two plus three is five.
ix) $x + 2 = 9$
x) The moon is made of green cheese.

4.4 LOGICAL CONNECTIVES

Till now, we have considered simple or primary statements which are declarative sentences, each of which cannot be expressed as a combination of more than one sentence. We often combine simple (primary) statements to form compound statements by using certain connecting words known as logical connectives. Primary statements are combined by means of connectives : *AND, OR, IF – THEN,* and *IF AND ONLY IF,* lastly *NOT.*

Now we will discuss in details compound statements and their truth values expressed in a tabular form, called Truth Table.

4.4.1 Negation

The denial of a statement P is called its negation and is written as **~P** and read as '**not P**'. Negation of any statement P is formed by writing "It is not the case that–'' or "It is false that–'' before P or inserting in P the word "not''.

Let us consider the statement

P : All integers are rational numbers .

The negation of this statement is :

~ P : It is not the case that all integers are rational numbers.

or

~ P : It is false that all integers are rational numbers.

or

~ P : It is not true that all integers are rational numbers.

Consider now the statement, $P : 7 > 9$

The negation of this statement is $\sim P : \sim(7 > 9)$ or $\sim P : (7 \leq 9)$

Truth Table of Negation : If the truth value of "P" is T, then the truth value of ~P is F. Also if the truth value of "P" is F, then the truth value of ~P is T.

The truth table of ~P is :

P	~P
T	F
F	T

Table 4.1

Illustrative Examples : Write the negation of the following statements :

i) $\sqrt{7}$ is a rational.

ii) $\sqrt{2}$ is not a complex number.

iii) Every natural number is greater than zero.

iv) All primes are odd.

v) All mathematicians are men .

Solution :

i) Let P denote the given statement i.e.,

P : $\sqrt{7}$ is a rational.

The negation of this statement is given by

~P : It is not the case that $\sqrt{7}$ is a rational.

or

~P : $\sqrt{7}$ is not a rational.

or

~P: It is false that $\sqrt{7}$ is rational .

ii) Let the given statement be denoted by P i.e.,

P : $\sqrt{2}$ is not a complex number.

The negation of this statement is given by

~P : $\sqrt{2}$ is a complex number.

or

~P : It is false that $\sqrt{2}$ is not a complex number.

iii) The negation of the given statement is :

It is false that every natural number is greater than 0.

or

There exists a natural number which is not greater than 0.

iv) The negation of the given statement is

There exists a prime which is not odd.

or

Some primes are not odd.

or

At least one prime is not odd.

v) The negation of the given statement is :

Some mathematicians are not men.

or

There exists a mathematician who is not man.

or

At least one mathematician is not man.

or

It is false that all mathematicians are men.

CHECK YOUR PROGRESS

Q.2. Write the negation of the following statements -

i) Bangalore is the capital of Karnataka.

ii) The Earth is round.

iii) The Sun is cold.

iv) Some even integers are prime.

v) Both the diagonals of a rectangle have the same length.

4.4.2 Conjunction

The conjunction of two statements P and Q is the statement "*P and Q*" which is denoted by $P \wedge Q$. P, Q are called the components of $P \wedge Q$.

Illustrative Examples :

i) The conjunction of the statements :

P : It is raining.

Q : 2 + 2 = 4 is

$P \wedge Q$: It is raining and 2 + 2 = 4 .

ii) Consider the statement :

The Earth is round and the Sun is cold.

Let Q : The Earth is round.

R : The Sun is cold.

Then $Q \wedge R$: The earth is round and the Sun is cold.

Truth table : The statement $P \wedge Q$ has the truth value T whenever both P and Q have the truth value T; Other wise it has the truth value F.

The truth table for conjunction as follows :

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 4.2

4.4.3 Disjunction

The disjunction of the two statements P and Q is the statement "*P or Q*", denoted by $P \vee Q$. P, Q are called the components of $P \vee Q$.

Illustration 3

i) Consider the compound statement

Two lines intersect at a point or they are parallel.

The component statements of this statement are :

P : Two lines intersect at a point.

Q : Two lines are parallel.

The given compound statement is the disjunction $P \vee Q$.

ii) Consider another statement

45 is a multiple of 4 or 6.

Its component statements are :

P : 45 is a multiple of 4 .

Q : 45 is a multiple of 6.

The given compound statement is the disjunction $P \vee Q$.

Truth table : The statement $P \vee Q$ has the truth value F only when both P and Q have the truth value F, $P \vee Q$ is true if either P is true or Q is true (or both Pand Q are true). Truth table for disjunction :

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 4.3

CHECK YOUR PROGRESS

Q.3. Write the following statements in symbolic form :

i) Pavan is rich and Raghav is not happy.

ii) Pavan is not rich and Raghav is happy.

iii) Naveen is poor but happy.

iv) Naveen is rich or unhappy

v) Naveen and Amal are both smart .

vi) It is not true that Naveen and Amal are both smart

vii) Naveen is poor or he is both rich and unhappy

vii) Naveen is neither rich nor happy.

Illustrative Examples : Write the component statements of the following compound statements and check whether the compound statement is true or false.

i) 50 is a multiple of both 2and 5.

ii) All living things have two legs and two eyes .

iii) Mumbai is the capital of Gujrat or Maharashtra.

iv) $\sqrt{2}$ is a rational number or an irrational number.

v) A rectangle is a quadrilatateral or a 5 sided polygon.

Solution :

i) The component statements of the given statement are

P : 50 is multiple of 2

Q : 50 is multiple of 5

We observe that both P and Q are true statements. Therefore, the compound statement $P \wedge Q$ is true.

ii) The component statements of the given statement are

P : All living things have two legs.

Q : All living things have two eyes. $\sqrt{2}$

We find that both P and Q are false statements. Therefore, the compound statement $P \wedge Q$ is false.

iii) The components statements of the given statement are

P : Mumbai is the capital of Gujrat.

Q : Mumbai is the capital of Maharashtra.

We find that P is false and Q is true. Therefore, the compound statement $P \vee Q$ is true.

iv) The component statements are

P : is a rational number.

Q : is an irrational number.

Clearly P is false and Q is true. Therefore, the compound statement $P \vee Q$ is true.

v) The component statement are

P : A rectangle is a quadrilateral.

Q : A rectangle is a 5 sided polygon.

We observe that P is true and Q is false. Therefore, the compound statement $P \vee Q$ is true.

CHECK YOUR PROGRESS

Q.4. Write the component statements of the following compound statements and find true values of the compound statements.

i) Delhi is in India and 2 + 2 = 4.

ii) Delhi is in England and 2 + 2 = 4.

iii) Delhi is in India and 2 + 2 = 5 .

iv) Delhi is in England and 2 + 2 = 5.

v) Square of an integer is positive or negative.

vi) The sky is blue and the grass is green .

vii) The earth is round or the sun is cold.

viiI) All rational numbers are real and all real numbers are complex.

ix) 25 is a multiple of 5 and 8.

x) 125 is a multiple of 7or 8.

4.4.4 Conditional Statements

If P and Q are any two statements, then the statement "if P, then Q", is called a conditional statement. It is denoted by $P \rightarrow Q$.

Example : Let P : Amulya works hard.

Q : Amulya will pass the examination.

Then $P \rightarrow Q$: If Amulya works hard, then he will pass the examination.

The statement P is called the ***antecedent*** and Q is called the ***consequent*** in $P \rightarrow Q$. The sign is called the sign of implication.

The conditional statement $P \rightarrow Q$ can also be read as :

i) P only if Q

ii) Q If P

iii) Q provided that P

iv) P is sufficient for Q

v) Q is necessary conditions for P

vi) P implies Q

vii) Q is implied by P.

Truth table : If the anticedent P is true and the consequent Q is false, then the conditional statement P→Q is false, otherwise it is true as given in the following table.

P	Q	P→Q
T	T	T
T	F	F
F	T	T
F	F	T

Table 4.4

Illustrative Examples :

1. Write each of the following statements in the form “If–then”
   i) You get job implies that your credentials are good.
   ii) A quadrilateral is a parallelogram if its diagonals bisect each other.
   iii) To get A+ in the class, it is necessary that you do all the excercises of the book.

Solution :

1. i) The given statement can be written as “If you get a job, then your credentials are good.”
   ii) The given statement can be written as-
   “If the diagonal of a quadrilateral bisect each other, then it is a parallelogram”.
   iii) The given statement can be written as
   “If you are to get A+ in the class, then you are to do all the exercises of the book”.
2. Write the following conditional statements in symbolic form and hence, find truth values.
   i) If 2 + 2 = 4, then Guwahati is in Assam
   ii) If 2 + 2 = 4, then Guwahati is in Bihar
   iii) if 2 + 2 = 5, then Guwahati is in Assam
   iv) If 2 + 2 = 5, then Guwahati is in Bihar

Solution : Let P : 2 + 2 = 4

Q : Guwahati is in Assam

R : 2 + 2 = 5

S : Guwahati is in Bihar

Then i) The given statement is $P \to Q$

As P and Q have truth values T each, so $P \to Q$ has truth value T, i.e., the given conditional statement is true.

ii) The given statement is $P \to S$

P	S	$P \to S$
T	F	F

So, the given statement is false.

iii) The given statement is $R \to Q$

R	Q	$R \to Q$
F	T	T

So, the given statements is true.

iv) The given statement is $R \to S$

R	S	$R \to S$
F	F	T

So, the given statement is true.

CHECK YOUR PROGRESS

Q.5. Write down the truth value of each of the following implication.

i) If 3 + 2 = 7, then Paris is the capital of india.

ii) If 3 + 4 = 7, then 3 > 7

iii) If 4 > 5, then 5 < 6.

iv) If 7 > 3, then 6 < 14

v) If 7 > 3, then 14 > 9.

4.4.5 Biconditional Statements

If P and Q are any two statements , then the statement 'P if and only if Q' is called a biconditional statement which is denoted by $P \leftrightarrow Q$. 'P if and only if Q' is also abbreviated as "P iff Q".

The biconditional 'P if and only if Q' is regarded as having the same meaning as 'if P, then Q and if Q, then P'. So the biconditional $P \leftrightarrow Q$ is the conjunction of the conditionals $P \rightarrow Q$ and $Q \rightarrow P$ i.e., $(P \rightarrow Q) \wedge (Q \rightarrow P)$ is same as $P \leftrightarrow Q$.

To find the truth table for $P \leftrightarrow Q$, we first find the truth table for $(P \rightarrow Q) \wedge (Q \rightarrow P)$:

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \wedge (Q \rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Table 4.5

Hence, the truth table for the biconditional $P \leftrightarrow Q$ is :

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 4.6

Thus, the biconditional $P \rightarrow Q$ is true only when both P, Q have identical truth values, otherwise it is false.

Examples of Biconditional statement :

1. A triangle is equilateral if and only if it is equiangular.
2. 8 > 4 if and only if 8 – 4 is positive.

NOTE

The statement $P \leftrightarrow Q$ can also be read as
a) Q if and only if P
b) P implies Q and Q implies P
c) P is necessary and sufficient condition for Q
d) Q is necessary and sufficient condition for P

3. 2 + 2 = 4 if and only if it is raining.
4. Two lines are parallel if and only if they have the same slope.

Illustrative Examples : Write the truth value of each of the following biconditional statements.

i) 4 > 2 if and only if 0 < 4 – 2 .
ii) 3 < 2 if and only if 2 < 1.
iii) 3 + 5 > 7 if and only if 4 + 6 < 10.
iv) 2 + 5 = 7 if and only if Guwahati is in Assam.

Solution : i) Let P : 4 > 2
Q : 0 < 4 – 2

Then, the given statement is $P \leftrightarrow Q$.

Clearly, P is true and Q is true and therefore, $P \leftrightarrow Q$ is true. Hence, the given statement is true, and its truth value is T.

ii) Let P : 3 < 2
Q : 2 < 1

Then, the given statement is $P \leftrightarrow Q$.

Clearly, P is false and Q is false and therefore, $P \leftrightarrow Q$ is true. Hence, the given statement is true, and its truth value is T.

iii) Let P : 3 + 5 > 7
Q : 4 + 6 < 10

Then, the given statement is $P \leftrightarrow Q$

Clearly, P is true and Q is false and therefore, $P \leftrightarrow Q$ is false. Hence, the given statement is false and therefore, its truth value is F.

iv) Let P : 2+5 =7
Q : Guwahati is in Assam

Then, the given statement is $P \leftrightarrow Q$. As P is false, Q is true, the given statement is false.

CHECK YOUR PROGRESS

Q.6. Write down the truth value of each of the following :

i) 3 + 5 = 8 if and only if 4 + 3 = 7.
ii) 4 is even if and only if 1 is prime.

iii) 6 is odd if and only if 2 is odd.

iv) 2 + 3 = 5 if and only if 3 > 5.

v) 4 + 3 = 8 if and only if 5 + 4 = 10.

vi) 2 < 3 if and only if 3 < 4.

4.5 CONVERSE, INVERSE AND CONTRA-POSITIVE OF A CONDITIONAL STATEMENT

The converse, inverse and contrapositive statements of a conditional statement $P \to Q$ are defined as follows :

Implication	$P \to Q$
i) Converse	$Q \to P$
ii) Inverse	$\sim P \to \sim Q$
iii) Contrapositive	$\sim Q \to \sim P$

Illustrative Examples :

1. Write down (i) the converse (ii) the inverse and (iii) the contrapositive of the following statement :

   If a quadrilateral ABCD is a square, then all the sides of quadrilateral ABCD are equal.

   Solution : Let P : Quadrilateral ABCD is a square

   Q : Sides of the quadrilateral ABCD are equal. Then the given statement is $P \to Q$

i) The converse is $Q \to P$, i.e.,if all the sides of a quadrilateral ABCD are equal, then quadrilateral ABCD is a square.

ii) The inverse is $\sim P \to \sim Q$, i.e., if a quadrilateral ABCD is not a square, then all the sides of quad. ABCD are not equal.

iii) The contrapositive is $\sim Q \to \sim P$, i.e., if all the sides of a quadrilateral ABCD are not equal, then quadrilateral ABCD is not a square.

CHECK YOUR PROGRESS

Q.7. Write down (i) the converse (ii) the inverse and (iii) the contrapositive of the following statements:

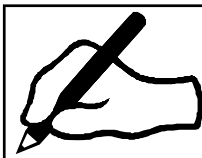
i) If a triangle is equilateral, it is isosceles.

ii) If x is prime number, then x is odd.

iii) If two lines are parallel,then they donot intersect in the same plane

iv) If a number is divisible by 9, then it is divisible by 3.

v) If a triangle is equilateral, then it is equiangular.



EXCERCISE

1. Find out which of the followig sentences are statements and which are not. Justify your answer.

   i) Every set is a finite set.

   ii) Are all circles round?

   iii) All triangles have three sides.

   iv) Is the earth round?

   v) Go !

2. Write the negation of the following statements:

   i) New Delhi is a city.

   ii) The number 2 is greater than 7.

   iii) The sum of 2 and 5 is 9.

3. Find the component statements of the following and express it in symbolic form :

   i) All integers are positive or negative.

   ii) All primes are even or odd.

   iii) 0 is a positive number or a negative number.

   iv) 24 is a multiple of 2, 4 and 8.

   v) 0 is less than every positive integer and every negative integer.

4. Write the components statements of each of the following statements and express if in symbolic form:
   i) If a natural number is odd, then its square is also odd.
   ii) If x = 4, then x^2 =16.
   iii) If ABCD is a parallelogram, then AB = CD.
   iv) If a number is divisible by 9, then it is divisible by 3.
   v) If a rectangle is a square, then all its four sides are equal.
5. Write down (i) the converse, (ii) the opposite and (iii) the contrapositive of the implications:
   i) If Mohan is a poet, then he is poor.
   ii) If she works, she will earn money.
   iii) If it snows, then they donot drive the car.
   iv) If x is less than zero, then x is not positive.
   v) If it is hot outside, then you feel thirsty.
6. Using truth table, find truth values of the following statements:
   i) It is false that two plus two equals four.
   ii) 2 + 2 = 4 and 4 + 4 = 9
   iii) 2 + 2 = 4 or 4 + 4 = 9
   iv) If 2 + 2 = 4, then 4 + 4 = 9
   v) 4 + 4 = 9 if and only if 8 + 8 = 18
7. Let P : She is tall
   Q : She is beautiful
   Write the following statements in sentences:
   i) $P \wedge Q$ ii) $P \wedge \sim Q$ iii) $\sim P \wedge \sim Q$
   iv) $\sim (P \vee \sim Q)$ v) $P \rightarrow (\sim Q)$ (vi) $\sim(\sim P \vee \sim Q)$

4.6 LET US SUM UP

- Logic is concerned with all kinds of reasoning, whether they are legal arguments or mathematical proofs or conclusions in a scientific theory based upon a set of hypotheses.
- Sentences are usually classified as declarative, exclamatory, interrogative and imperative. In our study of logic, we will confine

ourselves to the statements which are declarative sentences only and which are either true or false, but not both. A primary statement is a declarative sentence which cannot be further broken down or analyzed into simpler sentences.

- New statements can be formed from primary statements through the use of sentential connectives.The resulting statements are called *compound statements.*
- The sentential connectives are also called logical connectives. These connectives are: *negation, AND (conjunction), OR (disjunction), IF–THEN (conditional), IF AND ONLY IF (Bi-conditional), the symbols used are respectively.*
- Truth tables have already been introduced in the definitions of the connectives Our basic concern is to determine the truth value of a statement formula for each possible combination of the truth values of the component statements. A table showing all such truth values is called the truth table of the formula. we constructed the truth table for $\sim P$, $P \vee Q$, $P \wedge Q$, $P \rightarrow Q$ and $P \leftrightarrow Q$.Observe that if the truth values of the components are known,then the truth value of the resulting statement can be readily determined from the truth table by reading along the row which corresponds to the correct truth values of the component statements.
- The statement P is called the antecedent and Q is called the consequent in $P \rightarrow Q$.
- If P and Q are two statements , then the converse of the implication "if P, then Q" is "if Q, then P".

  The inverse of the implication "if P, then Q" is "if ~P, then ~Q".

  The contrapositive of the implication "if P, then Q" is "if ~Q, then ~P".

4.7 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 : Statement : i), iii), iv), vi), vii), viii) & x).

Ans. to Q. No. 2 : i) Bangalore is not the capital of Karnataka.

ii) The earth is not round.

iii) The sun is not cold.

iii) No even integer is prime.

iv) There is at least one rectangle whose both diagonals do not have the same length.

Ans. to Q. No. 3 : i) $P \wedge \sim Q$, where P : Pavan is rich
Q : Raghav is happy

ii) $\sim P \wedge Q$

iii) $\sim R \wedge H$, where R : Naveen is rich
H : Naveen is happy

iv) $R \vee \sim H$

v) $P \wedge Q$, where P : Naveen is smart
Q : Amal is smart

vi) $\sim(P \wedge Q)$

vii) $\sim R \vee (R \sim H)$, where R : Naveen is rich
R : Naveen is happy

viii) $\sim R \wedge \sim H$

Ans. to Q. No. 4 : i) P : Delhi is in india
Q : 2 + 2 = 4
The compound statement is true

ii) P : Delhi is in England
Q : 2 + 2 = 4
The compound statement is false.

iii) P : Delhi is in india
Q : 2 + 2 = 5
The compound statement is false

iv) P : Delhi is in England
Q : 2 + 2 = 5
The compound statement is false

v) P : Square of an integer is positive
Q : Square of an integer is negative
The compound statement is true.

vi) P : The sky is blue
Q : The grass is green
The compound statement is true.

vii) P : The earth is round
Q : The sun is cold
The compound statement is true.

viii) P : All rational numbers are real
Q : All real numbers are complex.
The compound statement is true.

ix) P : 25 is a multiple of 5
Q : 25 is a multiple of 8
The compound statement is false.

x) P : 125 is a multiple of 7
Q : 125 is a multiple of 8
The compound statement is false.

Ans. to Q. No. 5 : i) True, ii) False, iii) True, iv) False, v) True

Ans. to Q. No. 6 : i) True, ii) False, iii) True, iv) False, v) True, vi) True.

Ans. to Q. No. 7 :

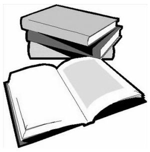
i) converse : If a triangle is isosceles, then it is equilateral.
inverse : If a triangle is not equilateral, then it is not isosceles.
contrapositive : If a triangle is not isosceles, then it is not equilateral.

ii) converse : If x is odd, then x is a prime.
inverse : If x is not prime, then ix s not odd.
contrapositive : If x is not odd, then x is not prime.

iii) converse : If two lines donot intersect in the same plane, then the lines are parallel.
inverse : If two lines are not parallel, then they intersect in the same plane.
contrapositive : If two lines intersect in the same plane, then the lines are not parallel.

iv) converse : If a number is divisible by 3, then it is divisible by 9.
inverse : If a number is not divisible by 9, then it is not divisible by 3.
contrapositive : If a number is not divisible by 3, then it is not divisible by 9.

v) converse : If a triangle is equiangular, then it is equilateral.
inverse: If a triangle is not equilateral, then it is not equiangular.
contrapositive : If a triangle is not equiangular, then it is not equilateral.



4.8 FURTHER READINGS

1. *Discrete Mathematical Structures with Application to Computer Science*, J. P Tremblay & R. Manohar.
2. *Discrete Structures and Graph Theory*, G. S. S. Bhishma Rao.

4.9 MODEL QUESTIONS

Q.1. Find out which of the following sentences are statements and which are not. Justify your answer.

i) The real number x is less than 2.

ii) All real numbers are complex numbers.

iii) Listen to me, Ravi !

Q.2. Find the component statements of the following and check whether they are true or not:

i) The sky is blue and the grass is green.

ii) The earth is round or the sun is cold.

iii) All rational numbers are real and all real numbers are complex

iv) 25 is a multiple of 5 and 8.

Q.3. Write the component statements of each of the following statements. Also, check whether the statements are true or not.

i) Sets A and B are equal if and only if ($A \subseteq B$ and $B \subseteq A$).

ii) $|a| < 2$ if and only if ($a > -2$ and $a < 2$)

iii) ΔABC is isosceles if and only if $\angle B = \angle C$.

iv) $7 < 5$ if and only if 7 is not a prime number.

v) ABC is a triangle if and only if $AB + BC > AC$.

Q.4. Write down (i) the converse, (ii) the opposite and (iii) the contrapositive of the implications :

i) If you live in Delhi, then you have winter cloths.

ii) If a quadrilateral is a parallelogram, then its diagonal bisect each other.

iii) If you access the website, then you pay a subscription fee.

iv) If you log on to the server, then you must have a passport.

v) If all the four sides of a rectangle are equal, then the rectangle is a square.

Q.5. If P is true and Q is false, then find truth values of

i) $P \wedge (\sim Q)$, ii) $\sim P \vee Q$, iii) $\sim P \rightarrow Q$,

iv) $P \rightarrow (\sim Q)$, v) $\sim (P \rightarrow Q)$, iv) $P \leftrightarrow Q$

UNIT 5 : TAUTOLOGY AND CONTRADICTION

UNIT STRUCTURE

5.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- define statement formulas or propositions
- define tautology and contradiction
- know about logical equivalence of two different statement formulas
- know about some important equivalence formulas
- learn about theory of inference.

5.2 INTRODUCTION

The notion of a statement has already been introduced in the previous unit. In this unit, we define statement formula and well-formed formula. Also we define tautology and contradiction of statement formulas and discuss equivalence of two statement formulas. Besides the theory of inference of statements will also be included on our discussion.

5.3 STATEMENT FORMULA OR PROPOSITION

Statements which donot contain any connective are called *simple or primary or atomic statements*. On the other hand, the statements which contain one or more primary statements and at least one connective are called ***composite or compound statements***.

For example, let P and Q be any two simple statements. some of the compound statements formed by P and Q are–

$\sim P$, $P \vee Q$, $(P \vee Q) \wedge (\sim P)$, $P \vee (\sim P)$, $(P \sim Q) \wedge P$.

The above compound statements are called statement formulas or prepositions derived from statement variables P and Q. Therefore P and Q are called components of the statement formulas. An arbitrary statement formula will be denoted by A(P,Q,...) or B(P,Q,...), etc.

A statement formula alone has no truth value. It has truth value only when the statement variables in the formula are replaced by definite statements and it depends on the truth values of the statements used in replacing the variables.

The truth table of a statement formula (Proposition) : Truth table has already been introduced in the definitions of the connectives. In general, if there are 'n' distinct components in a statement formula . We need to consider 2^n possible combinations of truth values in order to obtain the truth table.

For example, if any statement formula has two component statements namely P and Q, then 2^2 possible combinations of truth values must be considered.

Illustrative Examples :

1. Construct the truth table for $\sim P \wedge (\sim Q)$.

Solution : Truth Table :

P	Q	~Q	$P \wedge (\sim Q)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

2. Construct the truth table for $\sim P \vee \sim Q$.

   Solution : Truth Table :

P	Q	~P	~Q	~P ∨ ~Q
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

3. Construct the truth table for $P \rightarrow (Q \rightarrow R)$.

   Solution : P, Q and R are the three statement variables that occur in this formula $P \rightarrow (Q \rightarrow R)$. There are $2^3 = 8$ different sets of truth value assignments for the variables P, Q and R.

   The following table is the truth table for $P \rightarrow (Q \rightarrow R)$.

P	Q	R	Q → R	P → (Q → R)
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

CHECK YOUR PROGRESS

Q.1. Construct the truth tables for the following formulas

a) $\sim(\sim P \wedge \sim Q)$

b) $(\sim P \vee Q) \wedge (\sim Q \vee P)$

c) $(P \wedge Q)(P \vee Q)$

5.4 TAUTOLOGY

We have already defined truth table of a statement formula. In general, the final column of a given formula contains both T and F. There are some formulas whose truth values are always T or always F regardless of the truth value assignments to the variables. This situation occurs because of the special construction of these formulas.

Definition : *A statement formula which is true regardless of the truth values of the statements which replace the variables in it is called a universally valid formula or a tautology or a logical truth.*

A straight forward method to determine whether a given formula is a tautology is to constuct its truth table. In the table, if the column below the statement formula contains T only, then it is a tautology. The conjunction of two tautologies is also a tautology. Let us denote by A and B two statement formulas which are tautologies.If we assign any truth values of the variables of A and B, then the truth values of both A and B will be T. Thus the truth value of $A \wedge B$ will be T, so that $A \wedge B$ will be a tutology.

Illustrative Examples :

1. Verify whether $P \vee (\sim P)$ is a tautology.

Solution :

P	~P	$P \vee (\sim P)$
T	F	T
F	T	T

As the entries in the last column are T, the given formula is a tautology.

2. Verify whether (P ∨ Q)P is a tautology.

Solution :

P	Q	$P \vee Q$	$(P \vee Q) \rightarrow P$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

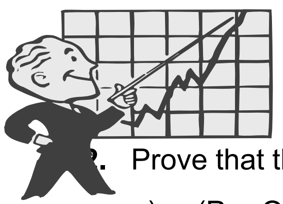
Since the entries in the last column of the truth table (PQ) P contain one false, the formula is not a tautology.

3. Verify whether $(P \wedge (P \leftrightarrow Q)) \rightarrow Q$ is a tautology.

Solution :

P	Q	$P \leftrightarrow Q$	$P \wedge (PQ)$	$(P \wedge (P \leftrightarrow Q)) \rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	T	F	T

As the entries in the last column are T, the given formula is a tautology.



CHECK YOUR PROGRESS

2. Prove that the following are tautologies (using truth tables):

a) $\sim(P \vee Q) \vee (\sim P \wedge Q) \vee P$

b) $(P \rightarrow Q) \leftrightarrow (\sim P \vee Q)$

c) $Q \vee (P \sim Q) \vee (\sim P \wedge \sim Q)$

5.5 CONTRADICTION

Definition : *A statement formula which is false regardless of the truth values of the statements which replace the variables in it is called a* ***contradiction***.

i.e, if each entry in the final column of the truth table of a statement formula is F only then it is called as contradiction.

Clearly, the negation of a contradiction is a tautology and vice-versa. We may call a statement formula which is a contradiction as identically false.

Illustrative Examples :

1. Verify that $P \wedge (\sim P)$ is a contradiction.

Solution :

P	$\sim P$	$P \wedge (\sim P)$
T	F	F
F	T	F

Since the last column has F only, the statement formula is a contradiction.

2. Verify the statement $(P \wedge Q) \wedge \sim(P \vee Q)$.

Solution :

P	Q	$P \wedge Q$	$P \vee Q$	$\sim(P \vee Q)$	$(P \wedge Q) \wedge \sim(P \vee Q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Since the truth value of $(P \wedge Q) \wedge \sim(P \vee Q)$ is F, for all values of P and Q, the proposition is a contradiction.

3. Prove that, if A(p, q, –) is a tautology, then ~A(p, q , –) is a contradiction and conversely.

Solution : Since a tautology is always true, the negation of a tautology is always false i.e is a contradiction and vice-versa.

5.6 LOGICAL EQUIVALENCE

Two statement formulas A(P, Q, ...) and B(P, Q, ...) are said to be logically equivalent or simply equivalent if they have identical truth tables. In other words, corresponding to identical truth values of P, Q, ... the truth values of A & B must be same. If A and B are equivalent, we shall write AB or $A \Leftrightarrow B$.

Illustrative Examples :

1. Prove that $P \vee Q \Leftrightarrow \sim(\sim P \wedge \sim Q)$.

Solution :

P	Q	$P \vee Q$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$	$\sim(\sim P \wedge \sim Q)$
T	T	T	F	F	F	T
T	F	T	F	T	F	T
F	T	T	T	F	F	T
F	F	F	T	T	T	F

The truth table shows that $P \vee Q$ and $\sim(\sim P \wedge \sim Q)$ have identical truth value column. So, $P \vee Q \Leftrightarrow \sim(\sim P \wedge \sim Q)$.

2. Prove that $P \rightarrow Q \Leftrightarrow (\sim P \vee Q)$.

Solution :

P	Q	$P \vee Q$	$\sim P$	$\sim P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Here, columns of $P \rightarrow Q$ and $(\sim P \vee Q)$ are identical.

Hence, $P \rightarrow Q \Leftrightarrow (\sim P \vee Q)$.

CHECK YOUR PROGRESS

Q.3. Show the following equivalences using truth table method:

a) $P \rightarrow (Q \vee R) \Leftrightarrow (P \rightarrow Q) \vee (P \rightarrow R)$

b) $\sim(P \rightarrow Q) \Leftrightarrow P \wedge \sim Q$

c) $P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$

d) $\sim(PQ) \rightarrow (\sim P \vee (\sim P \vee Q)) \Leftrightarrow (\sim P \vee Q)$

e) $(P \vee Q) \rightarrow (\sim P \wedge (\sim P \wedge Q)) \Leftrightarrow (\sim P \wedge \sim Q)$

f) $P \rightarrow Q \Leftrightarrow \sim Q \rightarrow \sim P$

5.7 EQUIVALENT FORMULAS

Using respective truth tables, we can prove the following equivalence:

a) Idempotent Laws : i) $P \vee P \Leftrightarrow P$ ii) $P \wedge P \Leftrightarrow P$

b) Associative Laws :

i) $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ ii) $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$

c) Commutative Laws :

i) $P \vee Q \Leftrightarrow Q \vee P$ ii) $P \wedge Q \Leftrightarrow Q \wedge P$

d) Distributive Laws :

i) $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

ii) $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

e) Absorption Laws :

i) $P \vee (P \wedge Q) \Leftrightarrow P$ ii) $P \wedge (P \vee Q) \Leftrightarrow P$

f) Demorgan's Laws:

i) $\sim(P \vee Q) \Leftrightarrow \sim P \wedge \sim Q$ ii) $\sim(P \wedge Q) \Leftrightarrow \sim P \vee \sim Q$

Some other important equivalence formulas:

i) $P \vee F \Leftrightarrow P$ ii) $P \wedge T \Leftrightarrow P$

iii) $P \vee T \Leftrightarrow T$ iv) $P \wedge F \Leftrightarrow F$

v) $P \vee \sim P \Leftrightarrow T$ vi) $P \wedge \sim P \Leftrightarrow F$

vii) $\sim\sim P = P$ viii) $\sim T = F$ ix) $\sim F = T$

Check yourself the above formulas as an exercise by truth table technique. Here, T and F respectively stands true statement and false statement.

Replacement Process : Consider the formula $A : P \rightarrow (Q \rightarrow R)$. The formula $Q \rightarrow R$ is a part of the formula. If we replace $Q \rightarrow R$ by an equivalent formula $\sim Q \vee R$ in A, we get another formula $B : P \rightarrow (\sim Q \vee R)$. we can easily verify that the formulas A and B are equivalent to each other.

This process of obtaining B from A is known as the replacement process. Using the laws stated in 5.7, we can also establish equivalence of statement formulas without using truth tables.

Illustrative Examples :

1. Prove that, $P \rightarrow (Q \rightarrow R) \Leftrightarrow P \rightarrow (\sim Q \vee R) \Leftrightarrow (P \wedge Q) \rightarrow R$.

Solution : We know that $Q \rightarrow R \Leftrightarrow \sim Q \vee R$

[see illustrative ex. 2 of 5.6]

Replacing $Q \rightarrow R$ by $\sim Q \vee R$, we get $P \rightarrow (\sim Q \vee R)$,

which is equivalent to $\sim P \vee (\sim Q \vee R)$ by the same rule.

Now, $\sim P \vee (\sim Q \vee R) \Leftrightarrow (\sim P \vee \sim Q) \vee R \Leftrightarrow \sim(P \wedge Q) \vee R \Leftrightarrow (P \wedge Q) \rightarrow R$, by associativity of $\vee$, Demorgan's law and the previously used rule.

2. Prove that, $(P \rightarrow Q) \wedge (R \rightarrow Q) \Leftrightarrow (P \wedge R) \rightarrow Q$

Solution : $(P \rightarrow Q) \wedge (R \rightarrow Q)$

$\Leftrightarrow (\sim P \vee Q) \wedge (\sim R \vee Q)$

$\Leftrightarrow (\sim P \wedge \sim R) \vee Q,$ [Distributive law]

$\Leftrightarrow \sim(P \vee R)\ Q$ [Distributive law]

$\Leftrightarrow (P \vee R) \rightarrow Q$

3. Prove that, $(\sim P \wedge (\sim Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$.

Solution : $(\sim P \wedge (\sim Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$

$\Leftrightarrow ((\sim P \wedge \sim Q) \wedge R) \vee ((Q \vee P) \wedge R)$

(Associative Law & distributive Law)

$\Leftrightarrow (\sim(P \vee Q) \wedge R) \vee ((Q \vee P) \wedge R)$ (Demorgan's Laws)

$\Leftrightarrow (\sim(P \vee Q) \vee (P \vee Q)) \wedge R$ (Distributive Law)

$\Leftrightarrow T \wedge R$ Since $\sim S \vee S \Leftrightarrow T$

$\Leftrightarrow R$ as $T \wedge R \Leftrightarrow R$

CHECK YOUR PROGRESS

Q.4. Prove that :

a) $(P \rightarrow Q) \wedge (R \rightarrow Q) \Leftrightarrow (P \vee R) \rightarrow Q$

b) $P \rightarrow (Q \rightarrow P) \Leftrightarrow \sim P \rightarrow (P \rightarrow Q)$

c) $\sim(P \leftrightarrow Q) \Leftrightarrow (P \vee Q) \wedge \sim(P \wedge Q)$

d) $\sim(P \leftrightarrow Q) \Leftrightarrow (P \wedge \sim Q) \vee (\sim P \wedge Q)$

Q.5. Show that P is equivalent to the following formulas :

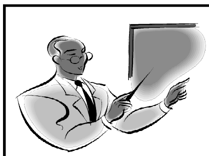
i) $\sim\sim P$ ii) $P \wedge P$ iii) $P \vee P$

iv) $P \vee (P \wedge Q)$ v) $P \wedge (P \vee Q)$

vi) $(P \wedge Q) \vee (P \wedge \sim Q)$ vii) $(P \vee Q) \wedge (P \vee \sim Q)$

5.8 TAUTOLOGICAL OR LOGICAL IMPLICATIONS

Definition : A statement A is said to tautologically or logically imply a statement B if and only if AB is a tautology. In this case,we write AB, read as "A tautologically implies B" or "A logically implies B".



LET US KNOW

i) $\Rightarrow$ is not a connective, $A \Rightarrow B$ is not a statement formula.

ii) Thus $A \Rightarrow B$ states that $A \rightarrow B$ is a tautology or A logically implies B.

iii) Clearly, $A \Rightarrow B$ guarantees that B has the truth value T whenever A has the truth value T.

iv) By constructing the truth tables of A and B, we can determine whether $A \Rightarrow B$.

v) In order to show any tautologically implications, it is sufficient to show that an assignment of the truth value T to the antecedent of the corresponding conditional leads to the truth value T for the consequent. This procedure ensures that the conditional becomes a tautology, thereby proving the tautological or logical implication.

vi) Another method to show $A \Rightarrow B$ is to assume that the consequent B has the value F and show that this assumption leads to A's having the value F. Then $A \rightarrow B$ must have the value T.

vii) $A \Leftrightarrow B$ if and only if $A \Rightarrow B$ and $B \Rightarrow A$ i.e, if each of two formulas A and B tautologically or logically implies the other, then A and B are equivalent.

Illustrative Examples :

1. Establish the following logical implication using the truth table :

$$(P \rightarrow (Q \rightarrow R)) \Rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow R)$$

Solution : We prove this by using the truth table for

$$(P \rightarrow (Q \rightarrow R)) \rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow R)$$

P	Q	R	$P \rightarrow Q$	$Q \rightarrow R$	$P \rightarrow R$	$P \rightarrow (Q \rightarrow R)$	$(P \rightarrow Q) \rightarrow (P \rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

As the columns of $P \rightarrow (Q \rightarrow R)$ and $(P \rightarrow Q) \rightarrow (P \rightarrow R)$ are identical, so $(P \rightarrow (Q \rightarrow R)) \rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow R)$ is a tautology.

Therefore $(P \rightarrow (Q \rightarrow R)) \Rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow R)$.

2. Show the following implication without constructing the truth tables:

$$\sim Q \wedge (P \rightarrow Q) \Rightarrow \sim P$$

Solution : To prove that $\sim Q \wedge (P \rightarrow Q) \Rightarrow \sim P$, it is enough to show that the assumption that $\sim Q \wedge (P \rightarrow Q)$ having the truth value T guarantees the truth value T for ~P.

Now assume that $\sim Q \wedge (P \rightarrow Q)$ has the truth value T. Then both ~Q and $P \rightarrow Q$ have the truth value T. Since ~Q has truth value T, Q has the truth value F. As Q has the truth value F and $P \rightarrow Q$ has the truth value T, it follows that the truth value of P is F and the truth value of ~P is T.

Thus we have prove that : $\sim Q \wedge (P \rightarrow Q) \Rightarrow \sim P$

Some Important Logical Implications :

1. $P \wedge Q \Rightarrow P$
2. $P \wedge Q \Rightarrow Q$
3. $P \Rightarrow P \vee Q$
4. $\sim P \Rightarrow P \rightarrow Q$
6. $Q \Rightarrow P \rightarrow Q$
5. $\sim(P \rightarrow Q) \Rightarrow P$
7. $\sim(P \rightarrow Q) \Rightarrow \sim Q$
8. $P \wedge (P \rightarrow Q) \Rightarrow Q$
11. $(P \rightarrow Q) \wedge (Q \rightarrow R) \Rightarrow P \rightarrow R$
12. $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \Rightarrow R$

Check yourself the above logical implications by using the truth table.

CHECK YOUR PROGRESS

Q.6. Show the following logical implications using the truth table:

a) $Q \Rightarrow P \rightarrow R$ b) $P\ Q \Rightarrow P \rightarrow Q$

Q.7. Show the following logical implications without constructing the truth tables :

a) $(P \vee Q) \wedge (\sim P) \Rightarrow Q$

b) $P \rightarrow Q \Rightarrow P \rightarrow (P \wedge Q)$

c) $(P \rightarrow Q) \rightarrow Q \Rightarrow P \vee Q$

d) $((P \vee \sim P) \rightarrow Q) \rightarrow ((P \vee \sim P) \rightarrow R) \Rightarrow (Q \rightarrow R)$

e) $(Q \rightarrow (P \wedge \sim P)) \rightarrow (R \rightarrow (P \wedge \sim P)) \Rightarrow (R \rightarrow Q)$

5.10 LOGICAL VALIDITY OF ARGUMENTS

One of the primary objectives of logic is to provide principles of reasoning for determining the validity of a conclusion subject to a given set of finite number of propositions, called ***premises***. In other words, logic provides us the principles of testing the validity of an argument. In this section we shall discuss validity of an argument, defined below, using truth tables.

Argument : Let $A_1, A_2, ...A_n$ be a finite number of statement formulas or propositions and A be another statement formula or proposition. Then the statement "$A_1, A_2,....A_n$ yields A" denoted by $A_1, A_2, ...A_n \vdash A$ is called an argument. $A_1, A_2, ...A_n$ are called premises and A is called the conclusion of the argument.

Example of Argument : Let us consider the following sentences :

"If a man is a bachelor, then he is unhappy.

If a man is unhappy, then he dies young.

Therefore, bachelors die young."

These three sentences form an argument, where the first two sentences are premises and the last sentence is the conclusion. To symbolise this argument, let us take

P : He is a bachelor

Q : He is unhappy

R : He dies young

Then the premises are $A_1 : P \to Q$, $A_2 : Q \to R$ and the conclusion is $A : P \to R$. In symbol, the given argument can be written as $A_1, A_2 \vdash A$.

Validity of an Argument : An argument $A_1, A_2, ..., A_n \vdash A$ ....(i) is called a *valid argument* if the conclusion A is true whenever all the premises $A_1, A_2, ..., A_n$ are simultaneously true.

NOTE

If $A_1, A_2, ...A_n \vdash A$ is a valid argument, it is said that the conclusion C follows logically from the set of premises $\{A_1, A_2, ..., A_n\}$

Now $A_1, A_2, ..., A_n$ are simultaneously true if and only if $A_1 \wedge A_2 \wedge ... \wedge A_n$ is true. In other words, the argument (i) is valid if and only if A is true whenever

$A_1 \wedge A_2 \wedge ... \wedge A_n$ is true i.e.

$(A_1 \wedge A_2 \wedge ... \wedge A_n) \to A$ is a tautology

or $A_1 \wedge A_2 \wedge ... \wedge A_n \Rightarrow A$

If an argument is not valid, it is called a *Fallacy*.

Truth table as an effective technique for testing validity of arguments : Let $P_1, P_2, ..., P_m$ be the primary statements occuring in one or more of the premises $A_1, A_2, ..., A_n$ and the conclusion A of an argument $A_1, A_2, ..., A_n \models A$.

To check the validity of the argument, we construct the truth table with columns for $A_1, A_2, ..., A_n$ and A. We find the rows where all the premises $A_1, A_2, ..., A_n$ have truth value T, simultaneously. If for every such row, A also has the truth value T, then we can easily show that $(A_1 \wedge A_2 \wedge ... \wedge A_n) \rightarrow A$ is a tautology, and hence, the argument is a valid argument. Alternatively, we can check those rows in which A has the truth value F. If in each of these roows, at least one of $A_1, A_2, ..., A_n$ has the truth value F, then again $(A_1 \wedge A_2 \wedge ... \wedge A_n) \Rightarrow A$ will be a tautology, and so, the argument will be valid.

Illustrative Examples : Examine validity of the arguments given below :

i) $P \rightarrow Q, \quad Q \rightarrow R \models P \rightarrow R$

ii) $P \rightarrow \sim Q, \quad Q \models \sim P$

iii) $P \rightarrow Q, \quad R \rightarrow \sim Q \models R \rightarrow \sim P$

iv) $P \rightarrow \sim Q, \quad \sim R \rightarrow \sim Q \models P \rightarrow \sim R$

[Note that in the above arguments, the statement formulas on the left of the symbol '$\models$' are the premises A_1, A_2, ... etc. and that on the right is the conclusion A.]

Solution : i) Truth Table :

P	Q	R	$P \rightarrow Q$	$Q \rightarrow R$	$P \rightarrow R$	Row
T	T	T	T	T	T	1
T	T	F	T	F	F	2
T	F	T	F	T	T	3
T	F	F	F	T	F	4
F	T	T	T	T	T	5
F	T	F	T	F	T	6
F	F	T	T	T	T	7
F	F	F	T	T	T	8

NOTE

The valid argument $P \rightarrow Q, Q \rightarrow R \models P \rightarrow R$ is called *law of syllogism*, where P, Q, R any three variable statements.

From the table, it is seen that both the premises are true in rows 1, 5, 7, 8 and the conclusion is also true in those rows. Hence, the given argument is valid.

Alternatively, the conclusion has the truth value F in rows 2 and 4 and in each of these two rows, at least one premise has the truth value F. So, the argument is valid.

ii) Truth Table :

P	Q	~Q	P → ~Q	~P
T	T	F	F	F
T	F	T	T	F
F	T	F	T	T
F	F	T	T	T

The table shows that both the premises have the truth value T only in row– 3 and the conclusion also has the truth value T in that row. So, the argument is valid.

iii) Truth table :

P	Q	R	~P	~Q	P → Q	R → ~Q	R → ~P	Row
T	T	T	F	F	T	F	F	1
T	T	F	F	F	T	T	T	2
T	F	T	F	T	F	T	F	3
T	F	F	F	T	F	T	T	4
F	T	T	T	F	T	F	T	5
F	T	F	T	F	T	T	T	6
F	F	T	T	T	T	T	T	7
F	F	F	T	T	T	T	T	8

The premises are simultaneously true in rows – 2, 6, 7, & 8 and the conclusion is also true in these rows. Hence the given argument is valid.

iv) Truth Table :

P	Q	R	~Q	~R	P → ~Q	~R → ~Q	P → ~R	Row
T	T	T	F	F	F	T	F	1
T	T	F	F	T	F	F	T	2
T	F	T	T	F	T	T	T	3
T	F	F	T	T	T	T	T	4
F	T	T	F	F	T	T	T	5
F	T	F	F	T	T	F	T	6
F	F	T	T	F	T	T	T	7
F	F	F	T	T	T	T	T	8

The third row shows that both the premises have truth value T, but the conclusion have the truth value F. hence, the given argument is not valid. It is a fallacy.

CHECK YOUR PROGRESS

Q.8. Examine validity of the following arguments :

a) $P, P \rightarrow Q \vdash Q$

b) $P \rightarrow Q, R \rightarrow \sim Q \vdash R \rightarrow \sim P$

c) $P \rightarrow \sim Q, Q \vdash \sim P$

d) $P \rightarrow \sim Q, \sim P \rightarrow R \vdash Q \rightarrow \sim R$

Q.9. Determine validity of the following arguments :

a) If it rains, I will stay at home. It did not rain. Therefore, I did not stay at home.

b) If it rains, I will stay at home. I did not stay at home. Therefore, it did not rain.

c) If I study, then I will not fail in mathematics. If I do not play cricket, then I will study. But I failed in mathematics. Therefore, I must have played cricket.

d) If I work hard, then I will get a job. If I get a job, then I will be happy. I will not be happy. Therefore, I will not work hard.

EXERCISE

1. From the formulas given below indicate which are tautologies or contradictions.

   a) $P \rightarrow (P \vee Q)$ b) $P \vee \sim(P \wedge Q)$

   c) $((\sim Q \wedge P) \wedge Q)$ d) $(P \wedge (P \rightarrow Q)) \rightarrow Q$

   e) $((\sim P \rightarrow Q) \rightarrow (Q \rightarrow P))$ f) $(P \wedge Q) \leftrightarrow P$

2. Show that the truth values of the following formulas is independent of its components.

   $(P \rightarrow Q) \leftrightarrow (\sim P \vee Q)$

3. Show the following logical implications constructing truth tables.
   a) $(P \wedge Q) \Rightarrow (P \rightarrow Q)$ b) $\sim(P \rightarrow Q) \Rightarrow P$
   c) $(P \rightarrow (Q \rightarrow R)) \Rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow R)$
4. Show the following equivalences.
   a) $P \rightarrow (Q \rightarrow P) \Leftrightarrow \sim P \rightarrow (P \rightarrow Q)$
   b) $\sim(P \wedge Q) \Leftrightarrow \sim P \vee \sim Q$
   c) $(P \rightarrow Q) \wedge (R \rightarrow Q) \Leftrightarrow (P \vee R) \rightarrow Q$
   d) $\sim(P \leftrightarrow Q) \Leftrightarrow (P \wedge \sim Q) \vee (\sim P \wedge Q)$
5. Show the following implications without constructing the truth tables.
   a) $P \rightarrow Q \Rightarrow P \rightarrow (P \wedge Q)$ b) $(P \rightarrow Q) \rightarrow Q \Rightarrow P \vee Q$

5.10 LET US SUM UP

- A statement formula is an expression which is a string consisting of (capital letters with or without subscripts), parentheses and connective symbols($\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$, ~), which produces a statement when the variables are replaced by statements.
- A statement formula which is true regardless of the truth values of the statements which replace the variables in it is called a universally valid formula or a tautology or a logical truth.
- A statement formula which is false regardless of the truth values of the statements which replaces the variables in it a contradiction.
- The statement formulas A and B are equivalent provided $A \leftrightarrow B$ is a tautology; and conversly, if $A \leftrightarrow B$ is a tautology, then A and B are equivalent. We shall represent the equivalence of A and B by writing "$A \Leftrightarrow B$" which is read as "A is equivalent to B." "$A \Leftrightarrow B$" is also denoted by '$A \equiv B$'
- A statement A is said is to tautologically imply a statement B if and only if $A \rightarrow B$ is a tautology. We shall denote this idea by $A \Rightarrow B$ which is read as "A logically implies B".
- For a given set of statement formulas A_1, A_2, ..., A_n and A; the statement "$A_1, A_2, ..., A_n$ yields A" is called an argument. It is denoted

by $A_1, A_2, ..., A_n \models A$. It is a valid argument if $(A_1 \wedge A_2 \wedge ... \wedge A_n)$, A is a tautology.

- If an argument is not valid, it is called a fallacy.

5.11 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 : a) The variable that occur in the formula are P and Q, so we have to consider 4 possible combinations of truth values of two statements P and Q.

P	Q	~P	~Q	~P ∧ ~Q	~(~P ∧ ~Q)
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

b) The variable are P and Q, clearly there are rows in the truth table of this formula.

P	Q	~P	~Q	~P ∨ Q	~Q ∨ P	(~P ∨ Q) ∧ (~Q ∨ P)
T	T	F	F	T	T	T
T	F	F	T	F	T	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

c)

P	Q	P ∧ Q	P ∨ Q	(P ∧ Q) → (P ∨ Q)
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Ans. to Q. No. 2 : b)

P	Q	~P	~P ∨ Q	P → Q	(P → Q) ↔ (~P ∨ Q)
T	T	F	T	T	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

All the entries in the last column are T, the given formula is a tautology.

Similarly, for (a) & (c) construct truth tables.

Ans. to Q. No. 3 : b)

P	Q	~Q	$P \rightarrow Q$	$\sim(P \rightarrow Q)$	$P \wedge \sim Q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

As $\sim(P \rightarrow Q)$ and $(P \wedge \sim Q)$ have identical truth columns, so $\sim(P \rightarrow Q) \Leftrightarrow P \wedge \sim Q$.

f)

P	Q	~P	~Q	$P \rightarrow Q$	$\sim Q \rightarrow \sim P$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Ans. to Q. No. 4 : a) We know that $P \rightarrow Q \Leftrightarrow \sim P \vee Q$

Similarly, $R \rightarrow Q$ by $\sim R \vee Q$

Now, $(P \rightarrow Q) \wedge (R \rightarrow Q) \Leftrightarrow (\sim P \vee Q) \wedge (\sim R \vee Q)$

$\Leftrightarrow (\sim P \wedge \sim R) \vee Q$ (By distributive law)

$\Leftrightarrow (\sim(P \vee R)) \vee Q$ (By De Morgan's law)

$\Leftrightarrow (P \vee R) \rightarrow Q$.

Similarly, you can prove (b), (c) & (d) by same process.

Ans. to Q. No. 5 : You can prove by truth table method.

Ans. to Q. No. 6 : a)

P	Q	$P \rightarrow Q$	$Q \rightarrow (P \rightarrow Q)$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

Since $Q \rightarrow (P \rightarrow Q)$ is a tautology, Therefore, $Q \Rightarrow (P \rightarrow Q)$.

Similarly, try to prove (b) by same process.

Ans. to Q. No. 7 : a) To prove that (P ∨ Q) ∧ (~P) ⇒ Q, it is enough to show that the assumption that (P ∨ Q) ∧ (~P) has the truth value T guarantees the truth value T for Q.

Now, assume that (P ∨ Q) ∧ (~P) has the truth value T. Then both (P ∨ Q) and ~P have the truth value T. Since ~P has truth value T, so P has truth value F. It follows the truth value of Q is T.

Thus we prove that (P ∨ Q) ∧ (~P) ⇒ Q.

Similarly, try to prove (b), (c), (d) & (e) by same process.

Ans. to Q. No. 8 : a) Valid, b) Valid, c) Valid, d) Fallacy

Ans. to Q. No. 9 : a) Take R : It rains
S : I stay at home

The argument is R → S, ~R |— ~S

Constructing the truth table, we can show that it is a fallacy.

b) In symbols of (a), the argument is R → S, ~ S |— ~R

It is a valid argument.

c) Take P : I study
Q : I fail in mathematics
R : I play cricket

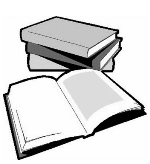
Then the argument is P → ~Q, ~R → P, Q |— R

It is a valid argument.

d) Take P : I work hard
Q : I will get a job
R : I will be happy

The argument is P → ~Q, Q → R, ~R |— ~P

It is a valid argument.



5.12 FURTHER READINGS

1. *Discrete Mathematical Structures with Application to Computer Science* by J. P. Tremblay & R. Manohar.
2. *Discrete Structures and Graph Theory* by G. S. S. Bhishma Rao.

5.13 MODEL QUESTIONS

Q.1. Construct the truth table for each of the following:

i) $(P \wedge Q) \rightarrow (P \vee Q)$

ii) $(P \wedge Q) \rightarrow \sim P$

iii) $(P \rightarrow Q) \leftrightarrow (\sim P \vee Q)$

Q.2. With the help of truth tables, prove the following :

i) $(P \rightarrow Q) \Leftrightarrow (\sim P \vee Q)$

ii) $(P \rightarrow Q) \Leftrightarrow (\sim Q \rightarrow \sim P)$

iii) $(P \leftrightarrow Q) \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$

Q.3. Show that the truth values of the following formulas are independent of their components .

a) $(P \wedge (P \rightarrow Q)) \rightarrow Q$

b) $(P \rightarrow Q) \leftrightarrow (\sim P \vee Q)$

c) $((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)$

d) $(P \leftrightarrow Q) \leftrightarrow ((P \wedge Q) \vee (\sim P \wedge \sim Q))$

Q.4. Given the truth values of P and Q as T and those of R and S as F, find the truth values of the following:

a) $(\sim(P \wedge Q) \vee \sim R) \vee ((Q \leftrightarrow \sim P) \rightarrow (R \vee \sim S))$

b) $(P \leftrightarrow R) \wedge (\sim Q \rightarrow S)$

c) $(P \vee (Q \rightarrow (R \wedge \sim P))) \leftrightarrow (Q \vee \sim S)$

Q.5. Examine validity of the following arguments :

a) $P \rightarrow Q, \sim Q \vdash P$

b) $P \rightarrow \sim Q, R \rightarrow Q, R \vdash \sim P$

c) $\sim Q, P \rightarrow Q \vdash \sim P$

d) $P \rightarrow (Q \rightarrow R), P \wedge Q \vdash R$

e) $P, P \rightarrow Q, Q \rightarrow R \vdash R$.

UNIT 6 : COUNTING PRINCIPLES

UNIT STRUCTURE

6.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- learn about basic counting principles
- describe Pigeonhole Principle and its applications.

6.2 INTRODUCTION

The study of arrangements of objects is an important part of discrete mathematics. Techniques of counting are important both in Mathematics and Computer Science, especially in probability theory and analysis of algorithms. In this unit we will introduce you to the basics of Counting and Pigeonhole principle.

6.3 BASIC COUNTING PRINCIPLES

The Rule of Sum : If a task can be performed in m ways, while another task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in $m + n$ ways.

Set theoretical version of the rule of sum : If A and B are disjoint sets ($A \cap B = \phi$) then $|A \cup B| = |A| + |B|$, where $|A|$ represent number of elements in A, etc.

More generally, if the sets $A_1, A_2, ..., A_n$ are pairwise disjoint, then $|A_1 \cup A_2 \cup A_3 \cup ..., \cup A_n| = |A_1| + |A_2| + |A_3| + ... + |A_n|$

Illustration Examples :

1. A scholarship is available, and the student to receive this scholarship must be chosen from the Mathematics, Computer Science, or the Engineering Department. How many different choices are there for this student scholarship if there are 38 qualified students from the Mathematics Department, 45 qualified students from the Computer Science Department and 27 qualified students from the Engineering Department?

Solution : The procedure of choosing a student from the Mathematics Department has 38 possible outcomes, the procedure of choosing a student from the Computer Science Department has 45 possible outcomes, and the procedure of choosing a student from the Engineering Department has 27 possible outcomes. Therefore, there are 38 + 45 + 27 = 110 possible choices for the student to be awarded the scholarship.

2. If a student has to choose a project from one of the 5 lists and the five lists contain 15, 20, 25, 10 and 12 possible projects respectively. How many possible projects are there to choose from?

Solution : The student can choose a project from the 1st list in 15 ways, from the 2nd list by 20 ways and so on. Using sum rule, there are 15 + 20 + 25 + 10 + 12 = 82 possible projects that can be chosen by a student.

3. In a class of 25 boys and 32 girls, find the number of ways of selecting one student as class representative.

Solution : Out of 25 boys, class representative can be selected in 25 ways and out of 32 girls, the same can be selected in 32 ways. Therefore, a class representative can be selected in 25 + 32 = 57 ways.

The Rule of Product : If a task can be performed in *m* ways and another independent task can be performed in *n* ways, then the combination of both tasks can be performed in *mn* ways.

Set theoretical version of the rule of product : Let A × B be the cartesian product of sets A and B.

Then, $|A \times B| = |A| \cdot |B|$

More generally, $|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|$

Illustration Examples :

1. A man has 10 shirts, 8 pairs of pants and 3 pairs of shoes. How many different outfits, consisting of one shirt, one pair of pants and one pair of shoes, are possible?

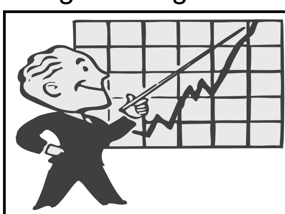
Solution : Choosing a shirt has 10 possible outcomes, choosing a pair of pants has 8 possible outcomes, and choosing a pair of shoes has 3 possible outcomes. So the number of different outfits is 10 x 8 x 3 = 240.

2. A student wishes to take a combination of 3 courses, one from each of the three Arts departments. There are 4 Economics, 3 History and 2 Political Science courses on offer. How many possible combinations are there?

Solution : The student has a choice of 4 courses in Economics and he can take any one of these 4 courses. Similarly, he can take any one of 3 courses in History and 2 courses in Politcal Science. Hence, required number of ways in which he can take a combination is 4 x 3 x 2 = 24.

3. How many different bits strings are there of length 5?

Solution : Each of the 5 bits can be chosen in 2 ways as each bit is either 0 or 1. Therefore, by product rule, there are $2^5 = 32$ different bit strings of length 5.



CHECK YOUR PROGRESS

Q.1. If a student is getting admission in 5 different Engineering Colleges and 4 different Medical Colleges, find the number of ways of choosing one of the above colleges?

Q.2. If A be the event of selecting a prime number less than 10 and B be the event of selecting an even number divisible by 4 and less than 10, find the number of ways of happening events A or B?

Q.3. For a set of six objective type questions which are either true or false, find the number of ways of answering all question.

Q.4. How many different license plates are available if each plate contains a sequence of two letters followed by four digits (and no sequence of letters are prohibitted).

6.4 PIGEONHOLE PRINCIPLE

Although it might look nothing much than common sense, the Pigeonhole principle is very useful in various types of problems. It allows us to sometimes draw quite unexpected conclusions in situations, when it even seems that we do not seem to have enough information. The Pigeonhole Principle is also known as Dirichlet Principle or Shoe Box Principle.

NOTE
The Pigeonhole Principle was first used by Dirichlet in Number Theory. The term *pigeonhole* actually refers to one of those old-fashioned writing desks with thin vertical wooden partitions in which to file letters.

Pigeonhole principle states– *If n + 1 or more objects are placed in n boxes, then at least one box contains more than one object.*

More generally we can say– If *n* pigeons are assigned to *m* pigeonholes then at least one pigeonhole contains two or more pigeons ($m<n$)

Proof : Let m pigeons holes be numbered with the numbers 1 through m. Starting with the pigeon 1, each pigeon is assigned in order to the pigeonholes as numbered before. Since $m < n$, i.e. the number of pigeonholes is less than the number of pigeons, $n – m$ pigeons are left without being assigned a pigeonhole. Thus, at least one pigeonhole will be assigned to a second pigeon.

Illustration Examples :

1. In any given set of 13 people at least two of them have their birthday during the same month.

Solution : Since 13 people can be thought of as the pigeons and 12 months of the year as the pigeonhole. Since 12 < 13, i.e. the number of pigeonholes is less than the number of pigeons, by Pigeonhole Principle, at least 2 of13 people have their birthday during the same month.

2. Let S be a set of eleven 2-digit numbers. Prove that S must have two elements whose digits have the same difference.

Solution : We consider the set S = {10, 14, 19, 22, 26, 28, 49, 53, 70, 90, 93}. The digits of the numbers 28 and 93 have the same difference: 8 – 2 = 6, 9 – 3 = 6. The digits of a two-digit number can have 10 possible differences (from 0 to 9). These possible differences can be considered as

10 pigeonholes and 11 numbers can be considered as pigeons. Since 10<11, by Pigeonhole principle in a list of 11 numbers there must be two with the same difference.

3. Prove that if seven distinct numbers are selected from {1, 2, ..., 11}, then two of these numbers sum to 12.

Solution : Let the pigeons be the numbers selected. We define six pigeonholes corresponding to the six sets: {1, 11}, {2, 10}, {3, 9}, {4, 8}, {5, 7}, {6}. When a number is selected, it gets placed into the pigeonhole corresponding to the set that contains it. Since seven numbers are selected and placed in six pigeonholes, some pigeonhole contains two numbers. By the way the pigeonholes were defined, these two numbers sum to 12.

4. Prove that if 11 integers are selected from among {1, 2, ..., 20}, then the selection includes integer a and b such that a - b = 2.

Solution : Let the pigeons be the 11 integers selected. We define 10 pigeonholes corresponding to the sets {3, 1}, {4, 2}, {7, 5}, {8, 6}, {11, 9}, {12, 10}, {15, 13}, {16, 14}, {19, 17}, {20, 18}. Place each integer selected into the pigeonhole corresponding to the set that contains it. Since 11 integers are selected and placed into 10 pigeonholes, some pigeonhole contains two numbers. By the way the pigeonholes were defined, these two integers differ by two.

If the number of pigeons is much larger than the number of pigeonholes, the above stated pigeonhole principle can be restated to give a stronger form

Extended Pigeonhole Principle : *If n pigeons are assigned to m pigeonholes, then one of the pigeonholes must contain at least* $\lfloor (n-1)/m \rfloor + 1$ *pigeons.*

Proof : Let us assume that none of the pigeonholes contain more than $\lfloor (n-1)/m \rfloor$ pigeons.

Then there are atmost $m\lfloor (n-1)/m \rfloor = n-1$ pigeons. This contradicts our assumption that there are n pigeons. Hence, one of the pigeonholes must contain at least $\lfloor (n-1)/m \rfloor + 1$ pigeons.

Illustration Examples :

1. If you have 5 rabbits sitting in 2 boxes, then there must be 3 or more rabbits in at least one of the boxes.

Solution : Here boxes can be considered as pigeonholes and thus m = 2 and each rabbit can be considered as pigeons, n = 5. By Pigeonhole principle, $\lfloor(5-1)/2\rfloor + 1 = 3$ i.e. there must be 3 or more rabbits in at least one of the boxes.

2. What is the minimum number of students required in a class to be sure that at least 5 will receive the same grade if there are 4 possible grades A, B, C and D?

Solution : The minimum number of students needed to ensure that at least 5 students receive the same grade is the smallest integer n such that $\lfloor(n-1)/4\rfloor + 1 = 5$. This gives n = 17 which is the minimum number of students needed to ensure that at least 5 students receive the same grade.

CHECK YOUR PROGRESS

Q.5. There are 3 men and 5 women in a party. Show that if these people are lined up in a row, at least two women will next to each other.

Q.6. Find the minimum number n of integers to be selected from S = {1, 2, 3, ..., ..., ..., 9} so that

a) the sum of two of the n integers is even,

b) the difference of two of the n integers is 5.

6.5 LET US SUM UP

- If a task can be performed in *m* ways, while another task can be performed in *n* ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in *m* + *n* ways.
- If a task can be performed in *m* ways and another independent task can be performed in *n* ways, then the combination of both tasks can be performed in *mn* ways.

- *If n + 1 or more objects are placed in n boxes, then at least one box contains more than one object. This principle is known as pigeonhole principle.*
- *If n pigeons are assigned to m pigeonholes, then one of the pigeonholes must contain at least* $\lfloor (n-1)/m \rfloor + 1$ *pigeons.*

6.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 : 9

Ans. to Q. No. 2 : 6

Ans. to Q. No. 3 : 64

Ans. to Q. No. 4 : 6760000

Ans. to Q. No. 5 : a) 35 b) 6

6.7 FURTHER READINGS

1. C. L. Liu, *Elements of Discrete Mathematics*, Tata McGraw-Hill Edition.
2. Seymour Lipschutz, Marc Lars Lipson, *Discrete Mathematics,* Tata McGraw-Hill Edition.

6.8 MODEL QUESTIONS

Q.1. If 20 candidates appear in a competitive examination then show that there exist at least two among them whose roll numbers differ by a multiple of 19.

Q.2. Determine the minimum number of elements to be selected from the set S = {1, 2, 3, 4, 5, 6, 7, 8, 9} so that sum of two of them is 10.

Q.3. Suppose 55 numbers are chosen in the range 1, 2, ..., 100. Show that there is a pair of numbers that differ by 9.

Q.4. Show that in any room of people who have been doing handshaking there will always be at least two people who have shaken hands the same number of times.

Q.5. If 9 books are to be kept in 4 shelves, there must be atleast one shelf which contains at least 3 books.

Q.6. Show that among any 4 numbers one can find 2 numbers so that their difference is divisible by 3.

Q.7. Given 12 different 2 digit numbers, show that one can choose two of them so that their difference is a two-digit number with identical first and second digit.

Q.8. Suppose that 17 people correspond by email, each one with all the rest. Each pair discusses one of three possible topics: politics, science, or religion. Show that there are at least three people who all correspond with each other about the same topic.

UNIT 7 : PERMUTATION AND COMBINATION

UNIT STRUCTURE

7.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- define permutation
- compute permutation of distinct objects
- describe factorial notation
- describe combination.

7.2 INTRODUCTION

In the previous unit, we have learnt about the counting principle. Here, in this unit, we shall learn some basic counting techniques. We will introduce you to permutations and combinations which are most important counting techniques widely used in various fields of modern science.

7.3 DEFINITION OF PERMUTATION

The word permutation means arrangement. For example, given 3 letters *a, b, c* suppose we arrange them taking 2 at a time. The various arrangements are *ab, ba, bc, cb, ac, ca*. Hence the number of arrangements of 3 things taken 2 at a time is 6 and this can be written as ${}^3P_2 = 6$.

NOTE
In permutations the order of arrangement is taken into account; when the order is changed, a different permutation is obtained.

Definition : The number of arrangements that $1 \leq r \leq n$ can be made out of n distinct things taking r at a time is called the number of permutations of n distinct things taken r at a time.

Notation : If n and r are positive integers such that , then the number of all permutations of n distinct things, taken r at a time is denoted by the symbol $P(n, r)$ or ${}^nP_r$.

Thus ${}^nP_r$ = Total number of permutations of n distinct things taken r at a time.

Example : Write down all the permutations of the vowels A, E, I, O, U in English alphabet taking 3 at a time and starting with E.

Solution: The permutations of vowels A, E, I, O, U taking three at a time and starting with E are EAI, EIA, EIO, EOI, EOU, EUO, EAO, EOA, EIU, EUI, EAU, EUA.

Clearly there are 12 permutations.

7.3.1 Permutation of Distinct Object

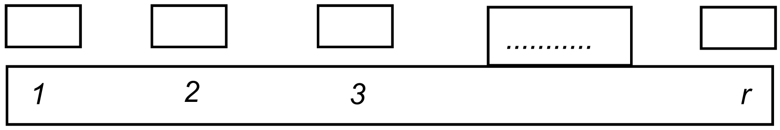
Theorem : Let r and n be positive integers such that $1 \leq r \leq n$. Then the number of all permutations of n distinct things taken r at a time is given by–

$$n(n-1)(n-2) \ldots (n-r+1)$$

i.e. ${}^nP_r = n(n-1)(n-2) \ldots (n-r+1)$

Proof : Let ${}^nP_r$ denote the number of permutations of n things taken r at a time.

Clearly the total number of permutations required is same as the number of possible ways of filling up r blank spaces by n things.



Let there be r blank spaces arranged in a row.

The first place can be filled up by any one of the n things in n ways. If the first place is filled up by any one of the n things, there will be $(n - 1)$ things remaining. Now the second place can be filled up by any one of the $(n - 1)$ remaining things i.e. the second can be filled up in $(n - 1)$ ways.

Hence the first two places can be together filled up in $n(n - 1)$ ways [See 6.3 Rule of Product].

Having filled up these two places, we have $(n - 2)$ things remaining with which we can fill up the third place. So the third place can be filled up by any one of these things in $(n - 2)$ ways.

Hence the first three places can be together filled in

$n(n - 1)(n - 2)$ ways.

Proceeding in this way, we find that the total number of ways of filling up the r spaces is

$n(n - 1)(n - 2)$ ... ... upto r factors

i.e. $n(n - 1)(n - 2) \ldots \ldots \ldots (n - (r - 1))$

$\therefore\ {}^{n}P_{r} = n(n - 1)(n - 2) \ldots \ldots \ldots (n - (r - 1))$

$= n(n - 1)(n - 2) \ldots \ldots \ldots (n - r + 1)$

Deduction : ${}^{n}P_{1} = n$, ${}^{n}P_{2} = n(n - 1)$, ${}^{n}P_{3} = n(n - 1)(n - 2)$, etc.

7.3.2 Factorial Notation

The continued product of first n natural numbers is called the "n factorial" and is denoted by $n!$ or $\underline{|n}$

i.e. $n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n - 1) \times n$

$1! = 1$

$2! = 1 \times 2$

$3! = 1 \times 2 \times 3$

$4! = 1 \times 2 \times 3 \times 4$ and so on.

We define 0! = 1

Thus we have, $n! = 1 \times 2 \times 3 \times 4 \times \dots \times (n-1) \times n$

$= [1 \times 2 \times 3 \times 4 \times \dots \times (n-1)]n$

$= [(n-1)!]n$

$n! = n\,[(n-1)!]$

For example, 8! 8(7!)

Example : Evaluate i) 6! ii) $\frac{9!}{7!}$

Solution : i) 6! = 1 x 2 x 3 x 4 x 5 x 6 = 720

ii) $\frac{9!}{7!} = \frac{9 \text{x} 8 \text{x} 7!}{7!} = 9 \times 8 = 72$

CHECK YOUR PROGRESS

Q.1. Evaluate 6! – 5!.

Q.2. Compute $\frac{n!}{r!(n-r)!}$, when $n = 4$ and $r = 2$

Q.3. If $\frac{1}{8!} + \frac{1}{9!} = \frac{x}{10!}$, find x.

7.3.3 Derivation of the Formula for ^nP_r

Theorem : Let r and n be positive integers such that $1 \le r \le n$.

Then $^nP_r = \frac{n!}{(n-r)!}$, $1 \le r \le n$

Proof : We know that $^nP_r = n(n-1)(n-2) \dots (n-r+1)$

Multiplying numerator and denomirator by

$(n-r)(n-r-1) \dots 3 \times 2 \times 1$, we get

$$^nP_r = \frac{n(n-1)(n-2)\dots(n-r+1)(n-r)(n-r-1)\dots 3 \times 2 \times 1}{(n-r)(n-r-1)\dots 3 \times 2 \times 1}$$

$$= \frac{n!}{(n-r)!}$$

Theorem : The number of all permutations of n distinct things, taken all at a time is n!

Proof : We have, ${}^nP_r = n(n-1)(n-2) \ldots (n-r+1)$

By putting $r = n$,

$${}^nP_n = n(n-1)(n-2) \ldots (n-n+1)$$
$$= n(n-1)(n-2) \ldots\ldots 1$$
$$= n!$$
$${}^nP_n = n!$$

Example : Evaluate ${}^8P_3$

Solution : ${}^8P_3 = \dfrac{8!}{(8-3)!} = \dfrac{8!}{5!} = \dfrac{(8 \times 7 \times 6) \times 5!}{5!}$

$= 8 \times 7 \times 6 = 336$

Example : If ${}^nP_4 = 360$, find the value of n.

Solution : ${}^nP_4 = 360$

$$\Rightarrow \frac{n!}{(n-4)!} = 6 \times 5 \times 4 \times 3$$

$$\Rightarrow \frac{n!}{(n-4)!} = \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!} = \frac{6!}{2!}$$

$\Rightarrow n! = 6!$

$\Rightarrow n = 6$

Example : In how many ways can five children stand in a queue?

Solution : The number of ways in which 5 persons can stand in a queue is same as the number of arrangements of 5 different things taken all at a time.

Hence the required number of ways = ${}^5P_5 = 5! = 120$

Example : Find the number of different 4-letter words with or without meanings, that can be formed from the letters of the word 'NUMBER'

Solution : There are 6 letters in the word 'NUMBER'.

So, the number of 4-letter words

= the number of arrangements of 6 distinct letters taken 4 at a time

= ${}^6P_4$

= 360.

7.3.4 Permutation of Objects not all Distinct

The number of mutually distinguishable permutations of n things, taken all at a time, of which p_1 are alike of one kind, p_2 are alike of second such that $p_1 + p_2 = n$, is $\frac{n!}{p_1!\,p_2!}$

We have a more general theorem–

The number of permutations of n objects, where p_1 objects are of one kind, p_2 are of second kind, ..., p_k are of k^{th} kind and the rest, if any, are all different kind is $\frac{n!}{p_1!\,p_2!\ldots p_k!}$

Example : How many permutations of the letters of the word 'APPLE' are there?

Solution : There are 5 letters, two of which are of the same kind. The rest are all different.

$\therefore$ Required number of permutations is $= \frac{5!}{2!\,1!\,1!\,1!}$

$$= \frac{5!}{2!} = \frac{120}{2} = 60$$

Example : How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places?

Solution : There are 4 odd digits 1, 1, 3, 3 and 4 odd places.

So odd digits can be arranged in odd places in $\frac{4!}{2!\,2!}$ ways

The remaining 3 even digits 2, 2, 4 can be arranged in 3 even places in $\frac{3!}{2!}$ ways.

Hence, the required number of numbers $= \frac{4!}{2!\,2!} \times \frac{3!}{2!}$

$$= 6 \times 3 = 18$$

7.3.5 Permutations When Objects Can Repeat

The number of permutations of n different things, taken r at a time, when each may be repeated any number of times in each arrangement, is n^r

Consider the following example:

In how many ways can 2 different balls be distributed among 3 boxes?

Let A and B be the 2 balls. The different ways are :

Box 1	*Box 2*	*Box 3*
A	*B*	
B	*A*	
	A	*R*
	R	*A*
A		*B*
R		*A*
		AR
AR		
	AB	

i.e. 9 ways. By formula $n^r = 3^2 = 9$ ways

Example : In how many ways can 5 different balls be distributed among 3 boxes?

Solution : There are 5 balls and each ball can be placed in 3 ways. So the total number of ways = $3^5 = 243$

Example : In how many ways can 3 prizes be distributed among 4 boys, when
i) no boy gets more than one prize?
ii) a boy may get any number of prizes?
iii) no boy gets all the prizes?

Solution :

i) The total number of ways is the number of arrangements of 4 taken 3 at a time.

So, the required number of ways = $^{4}P_{3}$ = 4! = 24

ii) The first prize can be given away in 4 ways as it may be given to anyone of the 4 boys.

The second prize can also be given away in 4 ways, since it may be obtained by the boy who has already received a prize.

Similarly, third prize can be given away in 4 ways.

Hence, the number of ways in which all the prizes can be given away = $4 \times 4 \times 4 = 4^3 = 64$

iii) Since any one of the 4 boys may get all the prizes, so the number of ways in which a boy gets all the 3 prizes = 4.

So, the number of ways in which a boy does not get all the prizes = 64 – 4 = 60

CHECK YOUR PROGRESS

Q.4. If $^{(n-1)}P_3 : {}^{n}P_4 = 1 : 9$, find n.

Q.5. Find the value of n such that $^{n}P_5 = 42\ {}^{n}P_3$, $n > 4$

Q.6. How many arrangements can be made with the letters of the word "MATHEMATICS"?

Q.7. How many 4 digit numbers canbe formed by using the digits 1, 2, 3, 4, 5, 6 if no digit is repeated? How many of these are even numbers?

7.4 COMBINATIONS

The word combination means selection. Suppose we are asked to make a selection of any two things from three things *a*, *b* and *c*. Then the different selections are *ab*, *bc*, *ac*. In selections, the order in which objects are selected is immaterial i.e. *ab* and *ba* denote the same selection. These selections are called combinations.

Definition : A selection of *r* things out of *n* distinct things is called a combination of *n* things taken *r* things at a time.

Notation : The number of all combinations of n distinct objects, taken r at a time is generally denoted by ${}^nC_r$ or C(n, r).

Thus ${}^nC_r$ = Number of ways of selecting r objects from n distinct objects.

Difference between Permutation and Combination :

1. In a combination only selection is made whereas in a permutation not only a selection is made but also an arrangement in a definite order is considered.

   i.e. in a combination, the ordering of the selected objects is immaterial whereas in a permutation, the ordering is essential.
2. Usually the number of permutation exceeds the number of combinations.

7.4.1 Derivation of the Formula for ${}^nC_r$

Theorem : The number of all combinations of n distinct objects, taken r at a time is given by ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$

Proof : Let the number of combinations of n distinct objects, taken r at a time be denoted by ${}^nC_r$.

Each of these combinations contains r things and if all these things are permuted among themselves, then we get ${}^rP_r = r!$ permutations.

The number of permutations obtained fromone combination = $r!$

Hence from all the ${}^nC_r$ combinations we get ${}^nC_r \times r!$ permutations.But this gives all the permutations of n things taken r at a time i.e. ${}^nP_r$.

Hence, ${}^nC_r \times r! = {}^nP_r$

$$\therefore\ {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{(n-r)!\,r!} \text{ as } {}^nP_r = \frac{n!}{(n-r)!}$$

Properties : 1) ${}^nC_n = 1$ 2) ${}^nC_0 = 1$

3) ${}^nC_r = {}^nC_{n-r}$ $0 \le r \le n$ 4) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

NOTE

${}^nC_x = {}^nC_y$

$\Rightarrow x = y$

or $x + y = n$

Proof : 1) We know that ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$

Putting $r = n$, we have ${}^nC_n = \dfrac{n!}{(n-r)!\,r!} = \dfrac{n!}{0!\,n!} = 1$ [$\because 0! = 1$]

2) Putting r = 0, we have ${}^nC_0 = \dfrac{n!}{(n-n)!\,r!} = \dfrac{n!}{n!} = 1$

3) We have ${}^nC_{n-r} = \dfrac{n!}{(n-r)(n-(n-r))!} = \dfrac{n!}{(n-r)!\,r!} = {}^nC_r$

4) We have ${}^nC_r + {}^nC_{r-1}$

$$= \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-(r-1))!(r-1)!}$$

$$= \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r-1)!(r-1)!}$$

$$= \frac{n!}{(n-r)!\,r\{(r-1)!\}} + \frac{n!}{(n-r-1)\{(n-r)!(r-1)!\}}$$

$$= \frac{n!}{(n-r)!(r-1)!}\left\{\frac{1}{r} + \frac{1}{n-r+1}\right\}$$

$$= \frac{n!}{(n-r)!(r-1)!}\left\{\frac{n-r+1+r}{r(n-r+1)}\right\}$$

$$= \frac{n!}{(n-r)!(r-1)!}\left\{\frac{n+1}{r(n-r+1)}\right\}$$

$$= \frac{(n+1)n!}{(n-r+1)(n-r)!\,r(r-1)!}$$

$$= \frac{(n+1)!}{(n-r+1)!\,r!}$$

$$= \frac{(n+1)!}{(n+1-r)!\,r!}$$

$$= {}^{n+1}C_r$$

Example : Evaluate the following :

i) ${}^6C_3$ ii) $\sum_{n=1}^{5} {}^5C_r$

Solution : i) ${}^6C_3 = \dfrac{{}^6P_3}{3!} = \dfrac{6\times5\times4}{1\times2\times3} = 20$

ii) $\sum_{n=1}^{5} {}^5C_r = {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$

$= 5 + 10 + 10 + 5 + 1 = 31$

Example : Let *r* and *n* be positive integers such that . Then prove the following :

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$$

Solution : We have $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{\frac{n!}{(n-r)!r!}}{\frac{n!}{(n-r+1)!(r-1)!}}$

$$= \frac{n!}{(n-r)!r!} \times \frac{(r-1)!(n-r+1)}{n!}$$

$$= \frac{(r-1)!(n-r+1)\{(n-r)!\}}{r(r-1)!(n-r)!}$$

$$= \frac{(n-r+1)}{r}$$

Example : From a group of 15 cricket players, a team of 11 players is to be chosen. In how many ways this can be done?

Solution : There are 15 players in a group. We have to select 11 players from the group.

The required number of ways $= {}^{15}C_{11}$

$$= \frac{15 \times 14 \times 13 \times 12}{1 \times 2 \times 3 \times 4}$$

= 1365 ways

Example : How many triangles can be formed by joining the vertices of a hexagon?

Solution : There are 6 vertices of a hexagon.

One triangle is formed by selecting a group of 3 vertices from given 6 vertices.

This can be done in ${}^6C_3$ ways.

Number of triangles = ${}^6C_3 = \frac{6!}{3!3!} = 20$

Example : A class contains 12 boys and 10 girls. From the class 10 students are to be chosen for a competition under the condition that atleast 4 boys and atleast 4 girls must be represented. Two girls who won the prizes last year should be included. In how many ways can the selection be made?

Solution : There are 12 boys and 10 girls. From these we have to select 10 students.

Since two girls who won the prizes last year are to be included in every selection.

So, we have to select 8 students from 12 boys and 8 girls, choosing atleast 4 boys and atleast 2 girls. The selection can be formed by choosing

i) 6 boys and 2 girls

ii) 5 boys and 3 girls

iii) 4 boys and 4 girls

Required number of ways

$= ({}^{12}C_6 \times {}^{8}C_2) + ({}^{12}C_5 \times {}^{8}C_3) + ({}^{12}C_4 \times {}^{8}C_4)$

$= (924 \times 28) + (792 \times 56) + (495 \times 70)$

$= 25872 + 44352 + 34650$

$= 104874$

CHECK YOUR PROGRESS

Q.8. If ${}^{n}C_4 = {}^{n}C_6$, find ${}^{12}C_n$.

Q.9. How many different teams of 8, consisting of 5 boys and 3 girls can be made from 25 boys and 10 girls?

Q.10. How many different sections of 4 books can be made from 10 different books, if

i) there is no restriction

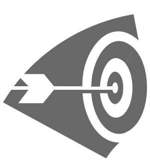
ii) two particular books are always selected;

iii) two particular books are never selected?

Q.11. In how many ways players for a cricket team can be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicket keepers? Assume that the team requires 5 batsmen, 3 all-rounder, 2 bowlers and 1 wicket keeper.

7.5 LET US SUM UP

- A permutation is an arrangement in a definite order of a number of objects taken some or all at a time.
- The number of permutations of n different things taken r at a time, where repetition is not allowed, is denoted by ${}^{n}P_{r}$ and is given by ${}^{n}P_{r} = \frac{n!}{(n-r)!}$, $1 \le r \le n$
- $n! = 1 \times 2 \times 3 \times ... \times n$
- $n! = n \times (n-1)!$
- The number of permutations of n different things, taken r at a time, where repetition is allowed, is n^r.
- The number of permutations of n objects taken all at a time, where p_1 objects are of first kind, p_2 objects are of the second kind, ..., p_k objects are of the k^{th} kind and rest, if any, are all different is $\frac{n!}{p_1!\,p_2!...p_k!}$
- The number of combinations of n different things taken r at a time, denoted by ${}^{n}C_{r}$, is given by ${}^{n}C_{r} = \frac{n!}{(n-r)!}$, $0 \le r \le n$.



7.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 : $6! - 5! = 720 - 120 = 600$

Ans. to Q. No. 2 : We have to evaluate $\frac{4!}{2!(4-2)!}$ (since $n = 4$ and $r = 2$)

$$= \frac{4!}{2!2!} = \frac{4 \times 3 \times 2!}{2!2!} = \frac{4 \times 3}{2!} = \frac{12}{1 \times 2} = 6$$

Ans. to Q. No. 3 : We have $\frac{1}{8!} + \frac{1}{9 \times 8!} = \frac{x}{10 \times 9 \times 8!}$

Therefore $1 + \frac{1}{9} = \frac{x}{10 \times 9}$ or $\frac{10}{9} = \frac{x}{10 \times 9}$

So, $x = 100$

Ans. to Q. No. 4 : $^{n-1}P_3 : {}^{n}P_4 = 1 : 9$

$\Rightarrow (n-1)(n-2)(n-3) : n(n-1)(n-2)(n-3) = 1 : 9$

$\Rightarrow 9(n-1)(n-2)(n-3) = n(n-1)(n-2)(n-3)$

$\Rightarrow n = 9$

Ans. to Q. No. 5 : Given that $^{n}P_5 = 42\,{}^{n}P_3$

or n(n–1)(n–2)(n–3)(n–4) = 42n(n–1)(n–2)

Since n > 4 so, n(n–1)(n–2) $\neq$ 0

Therefore, by dividing both sides by n(n–1)(n–2)

we get $(n-3)(n-4) = 42$

or $n^2 - 7n - 30 = 0$

or $n^2 - 10n + 3n - 30 = 0$

or $(n-10)(n+3) = 0$

or $n - 10 = 0$ or $n + 3 = 0$

or $n = 10$ or $n = -3$

As n cannot be negative, so, $n = 10$.

Ans. to Q. No. 6 : There are 11 letters in the word 'MATHEMATICS' of which two are M's, two are A's, two are T's and the rest are distinct.

$\therefore$ required number of arrangements = $\dfrac{11!}{2!\times 2!\times 2!}$ = 4989600

Ans. to Q. No. 7 : Total 4 digit numbers = $^{6}P_4 = \dfrac{6!}{4!}$ = 6 x 5 x 4 x 3 = 360

To get an even number, the units place should be filled up by 2 or 4 or 6. So, this place can be filled up in 3 ways. Then the remaining 3 places can be filled up in $^{5}P_3$ ways.

$\therefore$ Total 4 digit even numbers = 3 x $^{5}P_3$ = 3 x 5 x 4 x 3 = 180

Ans. to Q. No. 8 : $^{n}C_4 = {}^{n}C_6 \Rightarrow n = 4 + 6 = 10$

Now $^{12}C_n = {}^{12}C_{10} = {}^{12}C_{(12-10)} = {}^{12}C_2 = \dfrac{12\times 11}{1\times 2} = 66$

Ans. to Q. No. 9 : 5 boys out of 25 boys can be selected in $^{25}C_5$ ways.

3 girls out of 10 girls can be selected in $^{10}C_3$ ways.

$\therefore$ The required number of teams = $^{25}C_5 \times {}^{10}C_3$ = 6375600

Ans. to Q. No. 10 : i) The total number of ways of selecting 4 books out of 10 = $^{10}C_4 = \dfrac{10!}{4!6!}$ = 210

ii) If two particular books are always selected, then we are to select two books out of the remaining 8 books

$\therefore$ Required number of ways = $^8C_2 = \dfrac{8!}{2!6!} = 28$

iii) If two particular books are never selected, then we are to select four books out of the remaining 8 books

$\therefore$ Required number of ways = $^8C_4 = \dfrac{8!}{4!4!} = 70$

Ans. to Q. No. 11 : The selection of team is divided into 4 phases :

i) Selection of 5 batsmen out of 10. This can be done in $^{10}C_5$ ways.

ii) Selection of 3 all-rounders out of 5. This can be done in $^5C_3$ ways.

iii) Selection of 2 bowlers out of 8. This can be done in $^8C_2$ ways.

iv) Selection of 1 wicket keeper out of 2. This can be done in $^2C_1$ ways.

The team can be selected in $^{10}C_5 \times {}^5C_3 \times {}^8C2 \times {}^2C_1$ ways

$= 252 \times 10 \times 28 \times 2$ ways

$= 141120$ ways

7.7 FURTHER READINGS

1. C. L. Liu, *Elements of Discrete Mathematics*, Tata McGraw-Hill Edition.
2. Seymour Lipschutz, Marc Lars Lipson, *Discrete Mathematics,* Tata McGraw-Hill Edition.

7.8 MODEL QUESTIONS

Q.1. Evaluate : i) 8! ii) 4! – 3!

Q.2. If $\dfrac{1}{6!}+\dfrac{1}{7!}=\dfrac{x}{8!}$, find x.

Q.3. Evaluate $\dfrac{n!}{(n-r)!}$, when i) $n = 6$, $r = 2$; ii) $n = 9$, $r = 5$

Q.4. How many 3-digit numbers can be formed by using the digits 1 to 9 if no digit is repeated?

Q.5. How many 4-digit numbers are there with no digit repeated?

Q.6. How many 3-digit even numbers can be made using the digits 1, 2, 3, 4, 6, 7, if no digit is repeated?

Q.7. Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5 if no digit is repeated. How many of these will be even?

Q.8. From a committee of 8 persons, in how many ways can we choose a chairman and a vice chairman assuming one person can not hold more than one position?

Q.9. Find n if ${}^{n-1}P_3 : {}^{n}P_4 = 1 : 9$.

Q.10. Find r if i) ${}^{5}P_r = 2\ {}^{6}P_{r-1}$ ii) ${}^{5}P_r = {}^{6}P_{r-1}$

Q.11. How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once?

Q.12. How many words, with or without meaning can be made from the letters of the word MONDAY, assuming that no letter is repeated, if i) 4 letters are used at a time, ii) all letters are used at a time, iii) all letters are used but first letter is a vowel?

Q.13. If ${}^{n}C_8 = {}^{n}C_2$, find ${}^{n}C_2$.

Q.14. Determine n if (i) ${}^{2n}C_2 : {}^{n}C_2 = 12 : 1$ (ii) ${}^{2n}C_3 : {}^{n}C_3 = 11 : 1$

Q.15. How many chords can be drawn through 21 points on a circle?

Q.16. In how many ways can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls?

Q.17. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour.

Q.18. Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly one ace in each combination.

Q.19. In how many ways can one select a cricket team of eleven from 17 players in which only 5 players can bowl if each cricket team of 11 must include exactly 4 bowlers?

Q.20. A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected.

UNIT 8 : BASIC ALGEBRAIC STRUCTURE

UNIT STRUCTURE

8.1 LEARNING OBJECTIVES

After going through this unit, you will be able to :

- define a group and a semi-group
- learn about abelian groups, finite and infinite groups
- define order of a group
- learn about some elementary properties of groups
- define laws of indices of group-elements
- find examples of both abelian and non-abelian groups.

8.2 INTRODUCTION

The theory of groups plays an important role in present day mathematics and different branches of science, including computer science.

The structure of group is one of the simplest mathematical structures. It is the starting point in the study of various algebraic systems such as Rings, Fields, Vector Spaces, Linear operators, etc. In this present unit we shall discuss groups, semi-groups, subgroups, examples of these structures and their simple properties. We shall first define a **binary operation** on a non-empty set, a pre-requisite for defining a group.

8.3 BINARY OPERATION

A **binary operation** or **binary composition** denoted by $*$ in a set S is a mapping from S×S into S such that for all (a, b) $\in$ S×S, the image of (a, b) under the mapping $*$, denoted by $a * b$, belongs to S.

In short, an operation or composition $*$ defined on a non-empty set S is called a **binary operation** or **binary composition** if $a * b \in S$ for all $a, b \in S$. We shall henceforth use the symbol '$\forall$' to represent the phrase 'for all'. Thus, $*$ is a binary operation on S if $a * b \in S \; \forall \; a, b \in S$. If '$*$' is a binary operation on a set S, then S is said to be **closed** under '$*$' or the **closure property** holds in S.

NOTE : For the operation $*$ to be a binary operation in G, the **closure property** must hold in G, i.e., for all a, b $\in$ G, $a * b$ must belong to G.

Example1 : Take S = { –1, 1}, Then '×' is a binary operation on S, but; '+' is not a binary operation on S. To check it, we construct the following tables, showing addition and multiplication of the elements of S.

Table 1:

×	–1	1
–1	1	–1
1	–1	1

Table 2 :

+	–1	1
–1	–2	0
1	0	2

Table 1 shows that the resulting elements belong to S again. So, '×' is a binary operation on S. But table-2 shows that the resulting elements do not belong to S, and so, '+' is not a finary operation on S.

Example 2 : We know that sum of any two integers is again an intiger. So, '+' is a binary operation on Z. Similary '+' is a binary operation on Q, R, C also. '×' is also a binary operation on Z, Q, R, C. '÷' is not a binary operation on these number sets, since division by 0 is inadmissible.

8.4 DEFINITION OF GROUP

A non-empty set G together with a binary operation $*$, denoted by $(G, *)$ is called a group if the following postulates are satisfied.

1. Associativity : $a * (b * c) = (a * b) * c \;\forall\; a, b, c \in G$
2. Existence of identity : There exists an element $e \in G$, called identity element, such that
$$a * e = a = e * a \;\forall\; a \in G$$
3. Existence of inverse : For each $a \in G$, there exists an element, denoted by a^{-1} in G such that
$$a * a^{-1} = e = a^{-1} * a$$
a^{-1} is called the **inverse element** of a.

8.4.1 Abelian Group

A group $(G, *)$ is called an **Abelian group** or a **Commutative group** if the commutative law holds in G, i.e. $a * b = b * a \;\forall\; a, b \in G$ otherwise, a group is called a non-abelian or non-commutative group.

NOTE : Henceforth, unless specifically defined, we shall use the symbol '**.**' (dot) in place of '$*$' for a binary operation in a group, as '**.**' is more convenient to write rather than '$*$'. In fact, we can simply say that G is a group instead of saying $(G, *)$ is a group and we can write ab instead of $a * b$ if there is no likelihood of any confusion regarding the binary composition in the group.

8.4.2 Finite and Infinite Groups

A group $(G, *)$ is called a **finite** group if it has a finite number of elements, otherwise it is called an **infinite** group.

8.4.3 Order of a Group

The number of elements in a group $(G, *)$ is defined as the order of the group and is denoted by O (G) or | G |. If G is a finite group, then O (G) is a finite number. An infinite group is said to be of infinite order.

8.4.4 Semi Group

A non-empty set G together with a finary operation '$*$' is called a **semi group** if **associativity** holds in G, i.e,

$$a * (b * c) = (a * b) * c \;\forall\; a, b, c \in G$$

In otherwords, G is a semi-group if **closure** and **associativity** with respect to an operation '$*$' holds in G.

Similar to groups, we can also define abelian semi-group, finite and infinite semi-group, order of a semi-group. From definitions of Group and Semi-groups, it is obvious that **every group is a semi-group.** Later on, we shall find examples of semi-groups which are not groups. In otherwords, every semi-group may not be a group.

8.4.5 Examples of Groups and Semi Group

Example 1 : (Z, +) is a group, called **the groups of integers** under usual addition.

1. We know $a+b \in Z \;\forall\; a, b \in Z$ since sum of two integers is again an integer. Thus '+' is a binary operation in Z.
2. We know $a + (b+c) = (a+b) + c \;\forall\; a, b, c \in Z$, i.e., associativity holds in Z under addition.
3. $0 \in Z$ and for any $a \in Z$, we know
$$a + 0 = a = 0 + a$$
4. For $a \in Z$, $-a \in Z$, and we know
$$a + (-a) = 0 = (-a) + a$$
So, –a is the inverse element of a in Z. It is called the additive inverse of a.
5. For $a, b \in Z$, we know $a + b = b + a$, i.e., commutativity under addition holds in Z.

Hence (Z, +) is an infinite abelian group.

Example 2 : As in example 1, it can be easily shown that (Q, +), (R, +), (C, +) are all infinite abelian groups. These are respectively called **additive groups of rational numbers, real numbers and complex numbers**.

Example 3 : Let $Q^0 = Q - \{0\}$. Then (Q^0, x) is an infinite abelian group, called **the multiplicative group of non-zero rational numbers**.

1. Let a, b $\in Q^0$. Then a, b $\in$ Q and a $\neq$ 0, b $\neq$ 0. Clearly, ab $\in$ Q and ab $\neq$ 0. Hence ab $\in Q^0$, i.e., the closure property under multiplication holds in Q^0.
2. The associativity under multiplication holds in Q^0, since it holds in Q.
3. $1 \in Q^0$ and for any element $a \in Q^0$,

   $$a \times 1 = a = 1 \times a$$

   So, 1 is the multiplicative identity in Q^0.
4. $a \in Q^0 \Rightarrow a \in Q,\ a \neq 0$

   $$\Rightarrow \frac{1}{a} \in Q,\ \frac{1}{a} \neq 0$$

   $$\Rightarrow \frac{1}{a} \in Q^0$$

   Clearly $a \times \frac{1}{a} = 1 = \frac{1}{a} \times a$

   Hence, $\frac{1}{a}$ is the multiplicative inverse of a in Q^0.
5. Clearly, commutativity holds in Q^0. Thus (Q^0, x) is an infinite abelian group.

Example 4 : Let $R^0 = R - \{0\}$. Then as in example 3, it can be shown that (R^0, x) is an infinite abelian group. It is called the multiplicative group of non-zero real numbers.

Example 5 : Let $C^0 = \{x+iy : x, y \in R \text{ and } x+iy \neq 0\}$. Then ($C^0$, x) is an infinite abelian group.

1. Let a+ib, c+id $\in C^0$. Then a+ib $\neq$ 0, c+id $\neq$ 0. It can be shown that

   $$(a+ib)(c+id) = (ac-bd) + i(ad+bc) \in C^0$$

   So, closure property under multiplication holds in C^0.
2. Product of complex numbers obeys the associative law and so, the associativity also holds in C^0. [Prove yourselves]
3. 1 = 1 + i.0 $\in$ C and it is the multiplicative identity in C^0.
4. If a + ib $\in C^0$, a + ib $\neq$ 0 and hence

   $$\frac{1}{a+ib} = \frac{1}{a^2+b^2} - i.\frac{b}{a^2+b^2} \in C^0.$$ Clearly

$$(a+ib)\times\frac{1}{(a+ib)} = 1 = \frac{1}{(a+ib)} \times (a+ib)$$

Thus $\frac{1}{a+ib}$ is the multiplicative inverse of a + ib in C^0.

5. The commutative law (a+ib)(c+id) = (c+id)(a+ib) for a + ib, c + id $\in C^0$ is obvious.

   Hence (C^0, x) is an infinite abelian group.

   It is called the **multiplicative group of non-zero complex numbers**.

Example 6 : (Q^+, x), (R^+, x) are multiplicative groups of positive rational numbers and positive real numbers respectively. But (Z^+, x) is not a group, i.e., the set of positive integers is not a group under multiplication, because except –1 & 1, no element of Z^+ has multiplicative inverse. Clearly (Z^+, x) is a **semi-group** since colsure and associativity under multiplication holds in Z^+.

Example 7 : Let G = {1, w, w^2}, where w is an imaginary cube root of unity. It can be easily checked that G is a multiplicative group where 1 is the identity, inverse of w is w^2 and inverse of w^2 is w. It is a finite abelian group where 0(G) = 3.

Example 8 : Let G = {±1, ±i, ±j, ±k}. Let us define product in G as follows :

$1.1 = 1,\ (-1).(-1) = 1,\ i^2 = j^2 = k^2 = -1,$

$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$

It can be verified that G satisfies all the group postulates except commutativity, since ij ≠ ji. Hence G is a finite **non-abelian group**. This is called the **Quaturnion Group**.

Example 9 : Let G = {0, 1, 2, 3, 4, 5}. For any two elements a, b $\in$ G, let us define an operation denoted by ⊕ such that a ⊕ b = c, where c is the least non-negative integer obtained on dividing (a+b) by 6. The following table gives us the results when ⊕ is applied on G.

$\oplus$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Clearly, 1) closure property holds in G

2) associativity holds in G

3) 0 is the identity element.

4) additive inverses of 1, 2, 3, 4, 5 are 5, 4, 3, 2, 1 respectively.

5) the commutativity also holds.

Hence, (G, $\oplus$) is a finite abelian group.

It is called the **'Group of Residues Modulo 6'**.

In general, $Z_n = \{0, 1, 2, ..., n - 1\}$ is called the **'Group of Residues Modulo n'**, where the operation $\oplus$ is defined as

$$a \oplus b = c$$

where c is the least non-negative integer obtained on dividing a+b by c. It is a finite group, $0(Z_n) = n$. The operation $\oplus$ is called the operation of **'addition modulo n'**.

Example 10 : Let G be the set of all 2×2 non-singular matrices over real numbers, that is,

$$G = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} : x_i \in R \text{ and } \begin{vmatrix} x_1 & x_2 \\ x_3 & x_4 \end{vmatrix} \neq 0 \right\}$$

Then G is a group under matrix multiplication.

1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ be any two elements of G.

Then a, b, c, d, p, q, r, s $\in$ R and $|A| \neq 0$, $|B| \neq 0$.

Now $AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}$

Clearly ap + br, aq + bs, cp + dr, cq + ds $\in$ R.

Also $|AB| = |A|.|B| \neq 0$ as $|A| \neq 0$, $|B| \neq 0$.

Hence $AB \in G$, i.e., closure property holds in G.

2. As matrix multiplication is associative, so the associativity holds in G.

3. $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in G$ as $|I| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

   For any $A \in G$, $A.I = A = I.A$. Hence I is the identity element in G.

4. Since $A \in G \Rightarrow |A| \neq 0$, so $A^{-1} = \frac{1}{|A|}.\text{adj } A \in G$ such that $A.A^{-1} = I = A^{-1}.A$.

   Thus every element of G has its inverse in G. Hence G is a group under matrix multiplication.

   Take $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$. Then

   $$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 16 & 19 \\ 36 & 43 \end{pmatrix}$$

   $$BA = \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 19 & 28 \\ 27 & 40 \end{pmatrix}$$

   So, $AB \neq BA$, i.e., the commutative law does not hold in G. Hence, G is a non-abelian infinite group.

8.4.6 Properties of Groups

We shall now prove some properties of groups and semi-groups. We shall not mention the binary operation in considering a group G for establishing these properties. It should be understood that when we write the product ab for two elements $a, b \in G$; there exists a binary operation between a and b.

Property 1 : The identity element in a group is unique.

Proof : Suppose a group G has two identities e and e′.

Taking e as identity, we get

$e'e = e' = ee'$ ....(1)

Again taking e' as the identity, we get

$ee' = e = e'e$ ....(2)

From (1) and (2) we get

$e = e.e' = e'$

Hence the identity in G is unique.

Property 2 : The inverse of an element in a group is unique.

Proof : Let G be a group and $a \in G$. Suppose b and c are two inverse elements of G.

Then $ab = e = ba$ ......(1)

$ac = e = ca$ ......(2)

where e is the identity in G.

Now $b = be$

$= b(ac)$, from (2)

$= (ba)c$, using associativity

$= ec$, from (1)

$= c$, using definition of identity.

Hence the inverse of a is unique.

Property 3 : In a group G, i) $(a^{-1})^{-1} = a \ \forall a \in G$

ii) $(ab)^{-1} = b^{-1}a^{-1} \ \forall a, b \in G$

Proof : Let e be the identity element in G.

i) Let $a \in G$. Then

$aa^{-1} = e = a^{-1}.a$ ......(1)

Again $a^{-1} \in G$. So,

$a^{-1}(a^{-1})^{-1} = e = (a^{-1})^{-1}a^{-1}$ ......(2)

Now $(a^{-1})^{-1} = e(a^{-1})^{-1}$

$= (aa^{-1})(a^{-1})^{-1}$, using (1)

$= a[a^{-1}(a^{-1})^{-1}]$, using associativity

$= ae$, using (2) $= a$

ii) Let $a, b \in G$. Then $a^{-1}, b^{-1} \in G$

Now $(ab)(b^{-1}a^{-1}) = [(ab)b^{-1}]a^{-1}$, using associativity

$= [a(bb^{-1})]a^{-1}$

$= (ae)a^{-1}$

$= aa^{-1} = e$

NOTE : If G be an abelian group, then $(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1}$. In otherwords, for a non-abelian group, we cannot write $(ab)^{-1} = a^{-1}b^{-1}$.

Similarly, $(b^{-1}a^{-1})(ab) = e$

Thus $(ab)(b^{-1}a^{-1}) = e = (b^{-1}a^{-1})(ab)$

Hence, $(ab)^{-1} = b^{-1}a^{-1}$.

Property 4 : The concellation laws hold in a group.

That is, if $a, b, c \in G$, then

i) $ab = ac \Rightarrow b = c$ [left cancellation law]

ii) $ba = ca \Rightarrow b = c$ [right cancellation law]

Proof : Let e be the identity in a group G and $a, b, c \in G$.

i) $ab = ac \Rightarrow a^{-1}(ab) = a^{-1}(ac)$

$\Rightarrow (a^{-1}a)b = (a^{-1}a)c$, associativity

$\Rightarrow eb = ec$

$\Rightarrow b = c$

ii) $ba = ca \Rightarrow (ba)a^{-1} = (ca)a^{-1}$

$\Rightarrow b(aa^{-1}) = c(aa^{-1})$

$\Rightarrow be = ce$

$\Rightarrow b = c$.

NOTE : In general, in a group G,
$ab = ca \not\Rightarrow b = c$.
This is due to the fact that we cannot change the order of ca as ac unless G is abelian. For an additive group (G, +) a^{-1} charges to $-a$, $(a^{-1})^{-1}$ changes to $-(-a)$, the cancellation law
'$ab = ac \Rightarrow b = c$'
changes to
'$a+b = a+c \Rightarrow b = c$', etc.

Property 5 : For elements a, b in a group G, the equations $ax = b$ and $ya = b$ have unique solutions in G.

Proof : $ax = b \Rightarrow a^{-1}(ax) = a^{-1}b$

$\Rightarrow (a^{-1}a)x = a^{-1}b$

$\Rightarrow ex = a^{-1}b$

$\Rightarrow x = a^{-1}b$,

which is a solution of the equation $ax = b$, since $a^{-1}b \in G$.

If possible, suppose x_1, x_2 are two solutions of $ax = b$ in G.

Then $ax_1 = b$, $ax_2 = b \Rightarrow ax_1 = ax_2$

$\Rightarrow x_1 = x_2$,

using the left cancellation law.

Hence the solution of the equation $ax = b$ is unique in G.

Similarly, $ya = b$ has the unique solution $y = ba^{-1}$ in G.

8.4.7 Laws of indices in a Group

Leg G be a group and $a \in G$.

We define i) $a^0 = e$, the identity of G

ii) $a^n = a.a.a. \ldots a$, n factors where $n \in N$

iii) $a^{-n} = a^{-1}.a^{-1}.a^{-1}. \ldots a^{-1}$, n factors where $n \in N$

It can be shown that for all $m, n \in Z$,

$a^{-n} = (a^n)^{-1} = (a^{-1})^n$,

$a^m.a^n = a^{m+n}$

and $(a^m)^n = a^{mn}$

These are called laws of indices for group elements.

ILLUSTRATIVE EXAMPLES

Example 1 : In 'Q⁺', the set of positive rational numbers, define an operation $*$ such that

$a * b =$ $a, b \in Q^+$.

Show that $(Q^+, *)$ is an infinite abelian group.

Solution :

1) $a, b \in Q^+ \Rightarrow \dfrac{ab}{2} \in Q^+$

$\Rightarrow a * b \in Q^+$

Hence $*$ is a binary operation in Q^+.

2) For $a, b, c \in Q^+$,

$$a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a\left(\frac{bc}{2}\right)}{2} = \frac{abc}{4}$$

$$(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{\left(\frac{ab}{2}\right)c}{2} = \frac{abc}{4}$$

Thus $a * (b * c) = (a * b) * c$, showing that the associativity holds in Q^+.

3) For $a \in Q^+$, $a * 2 = a = 2 * a$ and so 2 plays the role of the identity element in Q^+ under '$*$'.

4) $a \in Q^+ \Rightarrow \dfrac{4}{a} \in Q^+$, and

$$a * \frac{4}{a} = 2 = \frac{4}{a} * a$$

Hence $\frac{4}{a}$ is the inverse of a in Q^+, i.e. $a^{-1} = \frac{4}{a} \; \forall \; a \in Q^+$.

Thus, all the group postulates are satisfied. Moreover, the commutative law hold in Q^+, since

$a * b = \frac{ab}{2} = \frac{ba}{2} = b * a$.

Hence $(Q^+, *)$ is an infinite abelian group.

Example 2 : Show that the set E of all even integers is a group under addition.

Solution :

1) Let $a, b \in E$. Then $a = 2m$, $b = 2n$, where $m, n \in Z$.

   Now, $a + b = 2m + 2n = 2(m+n) = 2l \in E$, since $l = m+n \in Z$.

   Thus E is closed under addition.

2) The associativity holds in E, since it holds in Z.

3) $0 \in E$ and it is the additive identity in E.

4) $a = 2m \in E \Rightarrow -a = -2m = 2(-m) \in E$ and $a + (-a) = 0 = (-a) + a$.

   So, every element in E has its additive inverse in E.

   Hence (E, +) is a group.

Example 3 : Show that a group G is abelian if and only if

$$(ab)^2 = a^2b^2 \; \forall \; a, b \in G.$$

Solution : Let G be abelian and $a, b \in G$.

Then $(ab)^2 = (ab)(ab)$

$= a(ba)b$, using associativity

$= a(ab)b$, as G is abelian

$= (aa)(bb)$

$= a^2b^2$.

Conversely, suppose $(ab)^2 = a^2b^2 \; \forall \; a, b \in G$.

Then $(ab)(ab) = (ab)(bb)$

$\Rightarrow a(ba)b = a(ab)b$

$\Rightarrow (ba)b = (ab)b$, using left cancellation law

$\Rightarrow ba = ab$, using right cancellation law

Thus $ab = ba \; \forall \; a, b \in G$ which shows that G is abelian.

CHECK YOUR PROGRESS

Q.1. Show that the set of integers Z is an abelian group under the operation $*$ defined by $a * b = a + b + 1 \; \forall \; a, b \in Z$.

Q.2. Show that Z is not a group under the operation $*$ defined by $a * b = a - b \; \forall \; a, b \in G$.

Q.3. Show that the set M of all 2×2 matrices over integers is a semi-group but not a group under matrix multiplication.

Q.4. Show that cancellation laws may not hold in a semi-group.

8.5 LET US SUM UP

- An operation $*$ defined in a non-empty set S is called a binary operation if $a * b \in S$ for all $a, b \in S$.
- A non-empty set G together with a binary operation $*$ is called a group if the following postulates are satisfied :

  Associativity : $a * (b * c) = (a * b) * c \; \forall \; a, b, c \in G$

  Existence of identity : $e \in G$ such that for any $a \in G$, $a * e = a = e * a$

Existence of inverse : for every $a \in G$, there exists $a^{-1} \in G$ such that $a * a^{-1} = e = a^{-1} * a$

The group is denoted by $(G, *)$.

- $(G, *)$ is an abelian or commutative group if $a * b = b * a \; \forall \; a, b \in G$
- A group is called a **finite group** if it has finite number of elements, otherwise a group is called infinite.
- The number of elements in a group $(G, *)$ is called the **order** of G, denoted by 0(G) or |G|.
- A non-empty set G together with a binary operation $*$ is called a semi-group if the associativity holds in G, i.e.,

  $a * (b * c) = (a * b) * c \; \forall \; a, b, c \in G$
- Every group is a semi-group, but every semi-group may not be a group.
- The identity element in a group is unique.
- The inverse of an element in a group is unique.
- In a group G, $(a^{-1})^{-1} = a \; \forall \; a \in G$ and $(ab)^{-1} = b^{-1}a^{-1} \; \forall \; a, b \in G$.
- The cancellation laws hold in a group G, i.e., if $a, b, c \in G$, then

  $ab = ac \Rightarrow b = c$

  $ba = ca \Rightarrow b = c$

8.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 : We show that Z under the given operation satisfies all the group postulates.

1) For $a, b \in Z$, $a + b + 1 \in Z \Rightarrow a * b \in Z$.

   So $*$ is a binary operation in Z.

2) For $a, b, c \in Z$

   $a * (b * c) = a + (b * c) + 1 = a + (b + c + 1) = a + b + c + 2$

   $(a * b) * c = (a * b) + c + 1 = (a + b + 1) + c + 1 = a + b + c + 2$

   So, $a * (b * c) = (a * b) + c$ and thus the associativity holds.

3) $-1 \in Z$ and $a * (-1) = a + (-1) + 1 = a$,

   $(-1) * a = (-1) + a + 1 = a$

   Thus $a * (-1) = a = (-1) * a$, showing that -1 is the identity in Z.

4) $a \in Z \Rightarrow -(a+2) \in Z$ and $a * [-(a+2)] = a - (a+2) + 1 = -1$

and $[-(a+2)] * a = -(a+2) + a + 1 = -1$.

Hence, $a * [-(a+2)] = -1 = [-(a+2)] * a$

[showing that $a^{-1} = -(a+2)$]

5) For $a, b \in Z$, $a * b = a + b + 1 = b + a + 1 = b * a$, showing that commutativity holds.

Hence $(Z, *)$ is an abelian group.

Ans. to Q. No. 2 : The operation $*$ defined in Z is given by

$$a * b = a - b \ \forall\ a, b \in Z.$$

Now $2, 3, 4 \in Z$ and

$2 * (3 * 4) = 2 * (3-4) = 2 * (-1) = 2 - (-1) = 3$

$(2 * 3) * 4 = (2-3) * 4 = (-1) * 4 = -1 - 4 = -5$

Thus we get $2 * (3 * 4) \neq (2 * 3) * 4$, i.e., the associativity under $*$ does not hold in Z.

Hence Z cannot be a group under the given operation.

Ans. to Q. No. 3 : $M_2(Z) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z \right\}$

Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$, $C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$

be any three elements of $M_2(Z)$ where $a_i, b_i, c_i \in Z$

1) $AB = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix}$

$$= \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$$

where $\alpha_1 = a_1b_1 + a_2b_3 \in Z$, $\alpha_2 = a_1b_2 + a_2b_4 \in Z$

$\alpha_3 = a_3b_1 + a_4b_3 \in Z$, $\alpha_3 = a_3b_2 + a_4b_4 \in Z$

Since $a_i, b_i, c_i \in Z$.

Thus AB is again a 2×2 matrix over Z, that is $AB \in M_2(Z)$.

2) Since A, B, C are 2×2 matrices, clearly the products AB, BC exist and it can be easily shown that A(BC) = (AB)C.

Thus the associativity holds in $M_2(Z)$.

Hence $M_2(Z)$ is a semi-group under matrix multiplication.

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the multiplicative identity in $M_2(Z)$. But existence of inverse element for every element does not hold in $M_2(Z)$.

For example, $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \in M_2(Z)$

where $|A| = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$, and so A^{-1} does not exist.

Hence $M_2(Z)$ is not a group.

Ans. to Q. No. 4 : Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix}$.

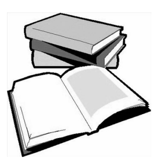
Then A, B, C $\in M_2(Z)$.

Now $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$, the zero matrix.

Also $AC = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$.

Thus AB = AC, but B $\neq$ C.

Hence cancellation laws may not hold in a semi-group.



8.7 FURTHER READINGS

1. *Modern Algebra* – S. Singh & Q. Zameeruddin, Vikas Pub. House Pvt. Ltc.
2. *A course in Abstract Algebra* – V. K. Khanna & S. K. Bhambri, Vikas Pub. House Pvt. Ltc.

8.8 MODEL QUESTIONS

Q.1. Show that (Q, +), (R, +), (P, +) are infinite abelian groups.

Q.2. Show that the set of non-zero real numbers R^0 is a group under multiplication.

Q.3. Show that the set of natural numbers N is a semi-group under the operations (i) addition, (ii) multiplication.

Q.4. Show that G = {–1, 1, –i, i} is an abelian group under multiplication. Write down 0(G).

Q.5. Let m be a positive integer greater than 1 and $mZ = \{0, \pm m, \pm 2m, \pm 3m, ...\}$ be the set of all integral multiples of m. Show that (mZ, +) is an abelian group.

Q.6. Give examples of :

i) a finite abelian group

ii) a finite non-abelian group

iii) an infinite abelian group

iv) an infinite non-abelian group.

Q.7. Let G = R – {–1}. Define an operation $*$ on G by $a * b = a+b+ab \ \forall a, b \in G$. show that (G, $*$) is an abelian group.

Q.8. Examine whether R is a group under the operation $*$ defined by $a * b = 2(a+b) \ \forall a, b \in R$.

Q.9. Show that if every element of a group is its own inverse, then the group is abelian.

Q.10. If in a group G, $a^2 = e \ \forall a \in G$, then show that G is abelian.

Q.11. Let G be any set having atleast two elements. For a, b $\in$ G, define $a * b = b$. Show that G is a semi-group under $*$ but not a group.

Q.12. Show that the set Q^+ of all positive rational numbers is a group under usual multiplication of numbers.

Q.13. Show that the set R^+ of all positive real numbers is a group under usual multiplication of numbers.

Q.14. Show that $M_2(R)$, the set of all 2×2 matrices over real numbers is an abelian group under the operation of matrix addition.

UNIT 9 : RING

UNIT STRUCTURE

9.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- define a ring
- know commutative ring
- know ring with unity
- know ring with zero divisors
- know elementary properties of a ring.

9.2 INTRODUCTION

In the preceding unit, are have discussed **Groups** – an algebraic system consisting of a non-empty set together with one binary operation. The most common of these groups are the groups of numbers (Z, +), (Q, +), (R, +), (C, +), (Q^0, x), (R^0, x), (C^0, x), etc. But we know that multiplication is also a binary operation in these sets, which is associative as well as distributive over addition. This leads us to the study of algebraic system with two binary operations. Such algebraic systems are **Rings**, **Integral Domains**, **Fields**, **Linear Spaces**, etc. In this unit we shall study Rings and elementary properties of a ring.

9.3 DEFINITION OF A RING

An algebraic system (R, +, **.**) consisting of a non-empty set R and two **binary operations**, denoted by '+' and '**.**' in R is called a **ring** if the following postudates are satisfied :

R_1 : (R, +) is an abelian group, i.e.,

i) $a + (b+c) = (a+b) + c \ \forall \ a, b, c \in R$

ii) There exists an element, donoted by 0, in R such that $a+0 = a = 0+a \ \forall \ a \in R$
'0' is called the **additive identity** or the **zero-element** of the ring.

iii) For every $a \in R$, there exists an element, denoted by –a, in R such that $a + (-a) = 0 = (-a) + a$
This element '–a' is called the **additive inverse** of a.

iv) $a+b = b+a \ \forall \ a, b \in R$

R_2 : (R, **.**) is a semi-group, i.e.,

v) $a.(b.c) = (a.b).c \ \forall \ a, b, c \in R$

R_3 : Multiplication is distributive over addition in R, i.e., for all a, b, $c \in R$

vi) $a.(b+c) = a.b + a.c$ (left distributive law)

vii) $(b+c).a = b.a + c.a$ (right distributive law)

If R has finite number of elements, then (R, +, **.**) is called a **finite ring**, otherwise it is called an **infinite ring**.

From group-properties we know that the identity element of a group is unique and every element of a group has unique inverse. If (R, +, **.**) is a ring, then by R_1, (R, +) is a group and hence the zero element, $0 \in R$ is unique. Similarly, for $a \in R$, the additive inverse '–a' is also unique.

Let us caution ourselves that whenever we use '+' and '**.**' for the two binary operations called 'addition' and 'multiplication' respectively in a ring (R, +, **.**), we should not confuse them with the arithmetic operations of addition and multiplication of real numbers. These operations may stand for addition and multiplication of ordered pairs, addition and multiplication of matrices, etc. Similarly, we should not confuse with the zero element of a ring with the real number 0 and the additive inverse –a of an element in a ring with the negative integer –a.

Basic Conventions :

1. If this is no scope for confusion regarding the binary operations, instead of writing '(R, + , **.**) is a ring', we shall simply write 'R is a ring'. It should be understood that + and **.** are the binary operations in the ring R.
2. The product of two elements a and b is a ring R will be simply written as ab, instead of a**.**b.
3. If a, b are two elements in a ring, then the element a + (–b) will be written as a – b.

9.3.1 Commutative Ring

If the commutativity under multiplication holds in a ring, that is, ab = ba $\forall$ a, b $\in$ R then R is called a **commutative ring**, otherwise R is called a **non-commutative ring**.

9.3.2 Ring with Unity

If in a ring R, there exists an element denoted by 1 such that a1 = a = 1a $\forall$ a $\in$ R, then R is called a **ring with unity**, and 1 is called the **multiplicative identity** or the **unity** in R. If 1 $\notin$ R, we say that R is a ring without unity.

We should note that 1 is just a symbol to denote the multiplicative identity of a ring R, we should not confuse it with the integer 1. Some authors also use the symbol 'e' for unity in a ring.

9.3.3 Ring with or without Zero-Divisors

Let R be a ring and a, b $\in$ R. If a $\neq$ 0, b $\neq$ 0 but ab = 0, then a and b are called **divisors of zero** or **zero-divisors** in R.

If there exists atleast one pair of zero-divisors in a ring R, then R is called **a ring with zero-divisors**.

R is called a **ring without zero-divisors** if for a, b $\in$ R, ab = 0 $\Rightarrow$ either a = 0 or b = 0.

9.3.4 Examples of Rings

Example 1 : (Z, +, **.**) is a commutative ring with unity and without zero-divisors.

We already know that (Z, +) is an abelian group.

Also a, b $\in$ Z $\Rightarrow$ ab $\in$ Z and so, multiplication is a binary operation in Z.

Clearly, for a, b, c $\in$ Z,

a(bc) = (ab)c,

a(b+c) = ab + ac,

(b+c)a = ba + ca

Also 1 $\in$ Z and a.1 = a = 1.a $\forall$ a $\in$ Z.

Again ab = ba $\forall$ a, b $\in$ Z.

Moreover, ab = 0 $\Rightarrow$ a = 0 or b = 0 is Z.

Hence Z is a commutative ring with unity 1 and without zero-divisors under the operations of usual addition and multiplication of numbers.

Example 2 : Similar to example 1, it can be easily shown that Q, R, C are commutative rings with unity and without zero-divisors under usual addition and multiplication.

Example 3 : If m be a positive integer greater than 1 and mZ = {0, ±m, ±2m, ±3m, ...}, then mZ is a commutative ring **without unity**, without zero-divisors.

Example 4 : Let M_2(Z) be the set of all 2×2 matrices over integers. Then it is a non-commutative ring with unity under addition and multiplication of matrices. It is a ring with zero-divisors.

It is a routine work to verify that R_1, R_2 and R_3 holds in M_2(Z) under the operations of matrix addition and multiplication, where $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the zero-element and the additive inverse of an element

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is } -A = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}.$$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the unity in $M_2(Z)$. Hence it is a ring with unity.

Now, let us take $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ in $M_2(Z)$. Then

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 10 & 13 \\ 22 & 29 \end{pmatrix} \text{ \&}$$

$$BA = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 11 & 16 \\ 19 & 28 \end{pmatrix}$$

and so, $AB \neq BA$.

Thus, commutativity under multiplication does not hold in $M_2(Z)$.
Hence $M_2(Z)$ is a non-commutative ring.

Again, let us take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ so that $A \neq 0$, $B \neq 0$.

But $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

Thus $A \neq 0$, $B \neq 0$ but $AB = 0$.

Hence A and B are zero-divisors, and so $M_2(Z)$ is a ring with zero-divisors.

Examples 5 : Let S be any set and P(S) be the power set of S. For A, B $\in$ P(S), let us define $A + B = (A–B) \cup (B–A)$, $AB = A \cap B$.

Then P(S) is a commutative ring with unity.

Clearly A+B $\in$ P(S), AB $\in$ P(S) $\forall$ A, B $\in$ P(S).

So + and **.** are binary operations in P(S).

It can be shown that for A, B, C $\in$ P(S)

A + (B+C) = (A+B) + C

$A + \phi = A = \phi + A$, and so ϕ is the zero-element in P(S).

A + A = ϕ = A + A), showing that A is its own additive inverse.

A + B = B + A

A(BC) = (AB)C

A(B+C) = AB + AC, (B+C)A = BA + CA

Moveover, AB = BA

AS = A $\cap$ S = A = S $\cap$ A = SA.

Hence P(S) is a commutative ring with unity S.

Examples 6 : Let R = {0, a, b, c} and let us define '+' and '**.**' in R by the following tables :

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

·	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	0	0	0

It can be easily checked from the above tables that R is a non-commutative ring without unity having zero-divisors.

Example 7 : In Z_n = {0, 1, 2, 3, ..., n–1}, let us define for a, b $\in Z_n$. a $\oplus$ b = c where c is the least non-negative remainder obtained on dividing a+b by c. Then from unit 8, we know that (Z_n, $\oplus$) is an abelian group.

Let us now define a $\otimes$ b = c, where c is the least non-negative remainder obtained on dividing ab by c. Then it can be proved that (Z_n, $\otimes$) is a semi-group and also the distributive laws of $\otimes$ over $\oplus$ holds in Z_n. Hence (Z_n, $\oplus$, $\otimes$) is a ring.

Moreover, 1 is the unity in Z_n and also a $\otimes$ b = b $\otimes$ a for a, b $\in Z_n$.

Hence Z is a commutative ring with unity. This ring is called the **'Ring of Integers Modulo n'**.

NOTE : Let us take the ring of integers modulo 6, Z_6 = {0, 1, 2, 3, 4, 5}. Clearly 2 $\otimes$ 3 = 0 but 2 $\neq$ 0, 3 $\neq$ 0, i.e., 2 and 3 are zero-divisors in Z_6. So, Z_6 is a ring with zero-divisors. In general, Z_n is a ring with or without zero-divisors according as n is a composite number or a prime number respectively.

9.4 PROPERTIES OF A RING

Let R be a ring. Then for all a, b $\in$ R the following properties hold

i) a0 = 0 = 0a

ii) a(–b) = (–a)b = –(ab)

NOTE : We already obtained laws of indices for group elements. Since for a ring (R, +, ·), both + and · are binary operations, that is, R is closed under + and ·, so the laws of indices for ring elements hold under certain restrictions. These are

na + ma = (n+m)a

na + nb = n(a+b)

where n, m are any integers.

If n, m are positive integers, then

$a^n.a^m = a^{n+m}$,

$(a^n)^m = a^{nm}$.

iii) (–a)(–b) = ab

iv) a(b–c) = ab – ac, (b–c)a = ba – ca

Proof :

i) For all a $\in$ R,

a0 = a(0+0)

$\Rightarrow$ a0 + 0 = a0 + a0, using definition of zero element and the left-distributive law.

$\Rightarrow$ 0 = a0, using the left-cancellation law in the group (R, +)

Similarly, 0a = 0

Hence a0 = 0 = 0a

ii) a[b+(–b)] = a0 = 0, by (i)

$\Rightarrow$ ab + a(–b) = 0, by left distributive law

$\Rightarrow$ ab + a(–b) = 0 = a(–b) + ab, as (R, +) is abelian

$\Rightarrow$ a(–b) = –(ab)

Similarly, (–a)b = –(ab)

Hence a(–b) = (–a)b = –(ab).

iii) (–a)(–b) = –[a(–b)], using (ii)

= –[–(ab)], again using (ii)

= ab [See note after property 4, 8.4.6]

iv) a(b–c) = a[b+(–c)]

= ab + a(–c)

= ab + [–(ac)]

= ab – ac.

Similarly, (b–c)a = ba – ca.

CHECK YOUR PROGRESS

Q.1. Give an example of a commutative ring with unity having 2 elements.

.

Q.2. Show that $R = \{a+ib : a, b \in Z\}$ is a ring with unity under usual addition and multiplication of complex numbers.

Q.3. Show that $Z_7 = \{0, 1, 2, 3, 4, 5, 6\}$ forms a ring under addition and multiplication modulo 7.

Q.4. Show that the cancellation laws hold in a commutative ring if and only if it has no zero-divisors.

Q.5. Prove that if R is a ring with unity, then the unity is unique.

Q.6. If the unity and the zero element of a ring R are equal, then show that $R = \{0\}$.

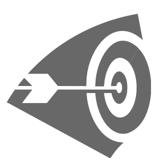
Q.7. If in a ring R, $a^2 = a \ \forall a \in R$, prove that :

i) $2a = 0 \ \forall a \in R$

ii) R is a commutative ring.

9.5 LET US SUM UP

- A non-empty set R together with two binary operations + and **.** is a ring if (R, +) is an abelian group, (R, **.**) is a semi-group and if the distributive laws of multiplication (**.**) holds over addition (+) is R.
- A ring R is called a commutative ring if ab = ba $\forall$ a, b $\in$ R.
- R is called a ring with unity if there exists an element 1 $\in$ R such that a1 = a = 1.a $\forall$ a $\in$ R.
- If there exist elements a, b in a ring R such that a $\neq$ 0, b $\neq$ 0 but ab = 0, then a, b are called zero-divisors and R is called a ring with zero-divisors.
- The following prosperties hold in a ring R :
  - i) a0 = 0 = 0a $\forall$ a $\in$ R
  - ii) a(–b) = (–a)b = –(ab) $\forall$ a, b $\in$ R
  - iii) (–a)(–b) = ab $\forall$ a, b $\in$ R
  - iv) a(b–c) = ab – ac, (b–c)a = ba – ca $\forall$ a, b $\in$ R.



9.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 : Take R = {0, 1} and define + and **.** is R as in the following tables.

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

It is now easy to verify that (R, +, **.**) is a commutative ring with unity 1.

Ans. to Q. No. 2 : R = {a+ib : a, b $\in$ Z}

Let a + ib, c + id $\in$ R where a, b, c, d $\in$ Z.

Then (a+ib) + (c+id) = (a+c) + i(b+d) $\in$ R, since a+c, b+d $\in$ Z.

Also (a+ib)(c+id) = (ac–bd) + i(ad+bc) $\in$ R,

since ac–bd, ad+bc $\in$ Z.

Hence, both addition and multiplication are binary operation is R.

If can be easily verified that the associative law under + and × holds in R, the commutative laws under + and × also holds in R, 0 = 0+i.0 is the zero element in R and 1 = 1+i.0 is the unity in R. Also the distributive laws of multiplication over addition holds in R. Hence, R is a commutative ring with unity under usual addition and multiplication.

Ans. to Q. No. 3 : Z_7 = {0, 1, 2, 3, 4, 5, 6}. The operations of addition and multiplication modulo 7 is shown in the tables below :

⊕	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	7	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

⊗	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

From the tables it can be easily verified that Z_7 is a commutative ring with unity 1.

Ans. to Q. No. 4 : Let R be a commutative ring without zero-divisors.

Let a, b, c $\in$ R such that a $\neq$ 0 and ab = ac.

Then ab – ac = 0

$\Rightarrow$ a(b–c) = 0, by the left distributive law

$\Rightarrow$ b – c = 0, since a $\neq$ 0 and R has no zero-divisors

$\Rightarrow$ b = c.

Thus a $\neq$ 0, ab = ac $\Rightarrow$ b = c Similarly, a $\neq$ 0 and ba = ca $\Rightarrow$ b = c.

Hence the cancellation laws hold in R.

Conversely, suppose the cancellation laws hold in R and let a, b $\in$ R such that ab = 0, a $\neq$ 0.

Then ab = a0, since a0 = 0 in R $\Rightarrow$ b = 0, by the left cancellation law.

Similarly ab = 0, b $\neq$ 0 $\Rightarrow$ a = 0.

Thus, ab = 0 $\Rightarrow$ a = 0 or b = 0 and so R has no zero-divisors.

Ans. to Q. No. 5 : If possible, let 1 and e be two unities in a ring R with unity.

Then 1e = e, taking 1 as unity.

Also $1e = 1$, taking e as unity.

Hence $e = 1$ and so the unity in R is unique.

Ans. to Q. No. 6 : Let R be a ring where $1 = 0$, i.e., the unity and the zero element are equal. Then $x \in R \Rightarrow x = 1x = ox = 0$. So, $R = \{0\}$.

Ans. to Q. No. 7 : Let R be a ring such that $a^2 = a \ \forall \ a \in R$.

i) $a \in R \Rightarrow a + a \in R$

$\Rightarrow (a+a)^2 = a + a$

$\Rightarrow (a+a)(a+a) = a + a$

$\Rightarrow (a+a)a + (a+a)a = a + a$, by the right distributive law

$\Rightarrow a^2 + a^2 + a^2 + a^2 = a + a$

$\Rightarrow (a+a) + (a+a) = (a+a) + 0$

$\Rightarrow a + a = 0$, by the left cancellation law in (R, +)

$\Rightarrow 2a = 0$.

ii) $a, b \in R \Rightarrow a + b \in R$

$\Rightarrow (a+b)^2 = a + b$

$\Rightarrow (a+b)(a+b) = a + b$

$\Rightarrow (a+b)a + (a+b)b = a + b$

$\Rightarrow a^2 + ba + ab + b^2 = a + b$

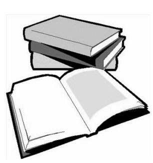
$\Rightarrow a + ba + ab + b = a + 0 + b$

$\Rightarrow ba + ab = 0$, using cancellation laws in (R, +)

$\Rightarrow ba + ab = ab + ab$, by (i)

$\Rightarrow ba = ab$, again by the right cancellation law.

Thus $ab = ba \ \forall \ a, b \in R$ and so R is a commutative ring.



9.7 FURTHER READINGS

1. *Modern Algebra* – S. Singh & Q. Zameeruddin.
2. *A course in Abstract Algebra* – V. K. Khanna & S. K. Bhambri.

9.8 MODEL QUESTIONS

Q.1. Show that $G = \{a + b\sqrt{3} : a, b \in Q\}$ is a commutative ring with unity under usual addition and multiplication of numbers.

Q.2. Define $\oplus$ an $*$ is Z as follows :

$a \oplus b = a + b - 1$, $a * b = a + b - ab$ for all a, b $\in$ Z.

Show that (Z, $\oplus$, $*$) is a commutative ring. Find the unity of this ring, if it exists.

Q.3. Let R × R = {(a, b) : a, b $\in$ R}. For (a, b), (c, d) $\in$ R×R, define addition and multiplication by (a, b) + (c, d) = (a + c, b + d) and

(a, b)(c, d) = (ac – bd, ad + bc)

Show that R × R is a commutative ring with unity.

Q.4. Give examples of :

i) A commutative ring with unity

ii) A non-commutative ring with unity

iii) A commutative ring without unity

iv) A non-commutative ring without unity.

Q.5. Give examples of :

i) A finite commutative ring with zero-divisors

ii) A finite commutative ring without zero-divisors.

Q/6. Show that $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in Z \right\}$ is a non-commutative ring without unity.

Q.7. Give an example of a non-commutative ring R such that $(ab)^2 = a^2b^2$ for all a, b $\in$ R.

Q.8. Prove that a ring R is a commutative ring if and only if $(a+b)^2 = a^2 + 2ab + b^2$ for all a, b $\in$ R.

Q.9. Let F be the set of all functions $f : R \to R$. For f, g $\in$ F, let us define f+g and fg as follows :

$(f+g)(x) = f(x) + g(x) \ \forall\ x \in R$

$(fg)(x) = f(x)g(x) \ \forall\ x \in R$.

Show that F is a commutative ring with unity under the above defined operations.

Q.10. Show that the set of even integers forms a commutative ring without unity under usual addition and multiplication.

UNIT 10 : INTEGRAL DOMAINS AND FIELDS

UNIT STRUCTURE

10.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- define an integral domain
- learn about the necessary and sufficient condition for a ring to be an I.D.
- define Division ring and Field
- learn about basic properties of I.D. and Field.

10.2 INTRODUCTION

In the preceding unit we defined zero-divisors in a ring and we come across rings with or without zero-divisors. This leads us to a special kind of ring called Integral domain. Similarly, existence of multiplicative inverses of non-zero elements in a commutative ring with unity leads to a special kind of ring called Field. In this unit we shall discuss Integral domain and Field and know some basic properties.

10.3 INTEGRAL DOMAIN (I.D.)

Before defining an I.D., let us recall zero-divisors in a ring discussed in the preceding unit.

A non-zero element ‘a’ of a ring R is called a zero-divisor if there exists another non-zero element ‘f’ in R such that ab = 0.

10.3.1 Definition of an I.D.

A commutative ring with unity in called on **Integral Domain (I.D.)** if it has no zero-divisors.

Alternatively, a commutative ring R with unity is called an I.D. if for all a, f $\in$ R, ab = 0 $\Rightarrow$ a = 0.

10.3.2 Examples of I.D.

Example 1 : Z, Q, R, C, i.e., the ring of integers, the ring of rational numbers, the ring of real numbers and the ring of complex numbers under usual addition and multiplication are all Integral domains since.

i) all these are commutative rings with unity 1,

and, ii) for any two elements a, b in each of these rings,
ab = 0 $\Rightarrow$ a = 0 or b = 0.
All these are infinite I.D’s.

Example 2 : Z_6 = {0, 1, 2, 3, 4, 5}, the ring of integers modulo 6 is not an I.D. even though it is a commutative ring with unity 1.

Because 2, 3 $\in$ Z_6, both non-zero; but 2 $\otimes$ 3 = 0, as 2.3 = 6 leaves the remainder 0 when divided by 6. Thus Z_6 has zero-divisors.

Example 3 : Z_p = {0, 1, 2, ... , p – 1}, the ring of integers modulo p is an I.D., if p is a prime.

We already know that Z_p is a commutative ring with unity 1.

Let a, b $\in Z_p$ such that a $\otimes$ b = 0. Then

$p \mid ab$

$\Rightarrow$ $p \mid a$ or $p \mid b$, since p is a prime

$\Rightarrow$ a = 0 or b = 0, since a, b < p

Thus a $\otimes$ b = 0 $\Rightarrow$ a = 0 or b = 0 and hence Z has no zero-divisors. Consequently, Z_p is an I.D.

NOTE : In some books, an I.D. is simply defined as a commutative ring without zero-divisors. The existence of unity is not taken into consideration. But most of the infinite commutative rings without zero divisors have unity. Also, every finite commutative ring without zero divisors contains the unity. Because of these two reasons, we have defined an I. D. to be a commutative ring with unity and without zero-divisors.

10.3.3 Necessary and Sufficient Condition for I.D.

Theorem 1 : A commutative ring with unity is an I.D. if and only if the cancellation laws hold in it.

Proof : Let R be a commutative ring with unity 1.

Suppose R is an I.D. Then R has no zero-divisors.

Let a, b, c $\in$ R such that ab = ac, a $\neq$ 0.

Then a (b – c) = 0 $\Rightarrow$ b – c = 0,

since a $\neq$ 0 and R has no zero-divisors

$\Rightarrow$ b = c.

Thus, ab = ac, a $\neq$ 0 $\Rightarrow$ b = c.

Similarly, ba = ca, a $\neq$ 0 $\Rightarrow$ b = c.

Hence the cancellation laws hold in R.

Conversely, let the cancellation laws hold in R.

Then for a, b, c $\in$ R and a $\neq$ 0.

ab = 0

$\Rightarrow$ ab = a0 [$\because$ a0 = 0]

$\Rightarrow$ b = 0, by the left cancellation law.

Thus ab = 0, a $\neq$ 0

$\Rightarrow$ b = 0

Similarly ab = 0, b $\neq$ 0

$\Rightarrow$ a = 0

Hence R has no zero-divisors and so, R is an I.D.

This completes the proof.

10.4 DIVISION RING AND FIELD

Let us consider the ring of integers Z and the ring of real numbers R under usual addition and multiplication. In Z, only –1 and 1 have multiplicative inverses. In R, every non-zero element has a multiplicative inverse, viz, if $a \in R$ and $a \neq 0$, then $\frac{1}{a} \in R$ and $\frac{1}{a}.a = 1 = a.\frac{1}{a}$, i.e., $\frac{1}{a}$ is the multiplicative inverse of 'a' in R. Thus, R is different from Z with respect to existence of multiplicative inverse of non-zero elements. Similarly, we shall find non-commutative ring with unity when every non-zero element has multiplicative inverse. This leads us to the definition of new classes of rings, such as Division Rings and Fields.

10.4.1 Definition and Examples of Units in a Ring

Definition : A non-zero element a of a ring R is called a **unit** if there exists on element b is R such that ab = ba = 1. In this ease, b is the multiplicative inverse of a in R.

In other words, a non-zero element of a ring is called a unit if its multiplicative inverse exists.

Examples : 1) In Z, only –1 and 1 are units.

2) In Q or R or C, every non-zero element is a unit.

3) In Z_7 = {0, 1, 2, 3, 4, 5, 6}, the ring of integers modulo 7 under addition and multiplication modulo 7, we can easily find

$1 \otimes 1=1,\ 2 \otimes 4=1,\ 3 \otimes 5 =1,\ 6 \otimes 6 =1.$

Hence, every non-zero element of Z_7 is a unit.

10.4.2 Division Ring

Definition : A ring with unity is called a **division ring** or a **skew field** if every non-zero element is a unit.

Alternatively, a ring with unity is called a division ring if every non-zero element has multiplicative inverse.

Examples : 1) The rings Q, R, C are division rings.

2) Z, 2Z etc. are not division rings.

3) Let R = {a + bi + cj + dk : a, b, c, d $\in$ R}

Where i, j, k are just three abstract symbols which can be multiplied by the following rules :

$i^2 = j^2 = k^2 = ijk = -1,$

$ij = -jk = k,\ jk = -kj = i,\ ki = -ik = j$

It can be proved that R is a ring under addition and multiplication with

0 = 0 + 0i + 0j + 0k as the zero element and

1 = 1 + 0i + 0j + 0k as the unity. It is a **non-commutative ring** since ij $\neq$ ji.

Now, let q = a + bi + cj + dk $\in$ R.

Then q $\neq$ 0 $\Rightarrow$ atleast one of a, b, c, d is non-zero.

$\Rightarrow a^2 + b^2 + c^2 + d^2 \neq 0$

So, $q^1 = \dfrac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2} \in R$

It can be easily shown that

$qq^1 = 1 = q^1q$, i.e., $q^{-1} = q^{-1}$.

Hence, every non-zero element of R is a unit an so R is a division ring.

This ring is called the **Ring of Real Quaternions**.

NOTE : An integral domain is generally denoted by D. Similarly, a field is generally denoted by F.

10.4.3 Fields

Definition : A commulative division ring is called a **field**.

Alternatively, a commutative ring with unity is called a field if every non-zero element has multiplicative inverse.

Obviously, the non-zero elements of a field form a multiplicative group. Hence, we may also define a field as follows :

A commutative ring R with unity is called a field if (R–{0}, **.**) is a group.

Examples : 1) Q, R, C are all fields

2) Z is not a field since non-zero integers except –1, 1 have no multiplicative inverses.

3) The ring of real quaternions $R = \{a + bi + cj + dk;\ a, b, c, d \in R\}$ is not a field, since it is non-commutative.

4) $Z_p = \{0, 1, 2, ..., p-1\}$, where p is a prime, is a field under addition and multiplication modulo p.

10.5 PROPERTIES OF I.D. AND FIELD

The basic properties of I.D. and field are the following :

1) The cancellation laws hold in an I.D.
2) Every field is an I.D. and,
3) Every finite I.D. is a field.

1) has been proved in 10.3.3 Theorem 1. We shall now prove 2) and 3) in the following theorems.

Theorem 2 : Every field is an I.D.

Proof : Let F be a field. Then F is a commutative ring with unity and every non-zero element of F has multiplicative inverse.

Let $a, b \in F$ such that $a \neq 0$. Then $a^{-1} \in F$.

Now $ab = 0 \Rightarrow a^{-1}(ab) = a^{-1}0$

$\Rightarrow (a^{-1}a)\, b = 0$

$\Rightarrow 1b = 0$

$\Rightarrow b = 0$

Thus $a \neq 0$, $ab = 0 \Rightarrow b = 0$

Similarly $b \neq 0$, $ab = 0 \Rightarrow a = 0$

Hence F has no zero divisors and so F is an I.D. This completes the proof.

Theorem 3 : Every finite integral domain is a field.

Proof : Let R be a finite I.D. and so R is a commutative ring with unity. To prove that R is a field, it is enough to show that every non-zero element of R has multiplication inverse.

Since R is finite, we can take $R = \{0, 1, a_1, a_2 ..., a_n\}$.

Let a be a non-zero element of R, i.e., $a \neq 0$. If a = 1, $a^{-1} = 1$.

So, We take $a \neq 1$. By closure property.

$R' = \{a1, aa_1, aa_2, ... aa_n) \subset R$

Since R has non-zero divisors, all the elements of R′ are non-zero. Moreover, the cancellation laws hold in R and so for $i \neq j$,

$aa_i = aa_j \Rightarrow a_i = a_j$, a contradiction.

This shows that all the elements of R′ are distinct.

Thus R′ has (n+1) distinct non-zero elements and so

$R' = R - \{0\} = \{1, a_1, a_2, ..., a_n\}$

Hence, $1 \in R' \Rightarrow 1 = aa_i$ for some $a_i \in R$.

The commutativity of R shows that $aa_i = 1 = a_ia$

Hence a^{-1} exists and $a^{-1} = a_i$.

Thus every non-zero element of R has multiplicative inverse, which completes the proof.

CHECK YOUR PROGRESS

Q.1. Give an example to show that every I.D. is not a field.

Q.2. If p is a prime, prove that Z_p, the ring of integers modulo p is a field.

Q.3. Give an example to show that Z_n, the ring of integers modulo n is not a field if n is a composite number.

Q.4. Show that a field has no zero-divisors.

10.6 LET US SUM UP

- An I. D. is a commutative ring with unity and without zero-divisors.
- A commutative ring with unity is an I.D. if and only if the cancellation laws hold in it.
- A non-zero element of a ring is called a **unit** if its multiplicative inverse exists.
- A ring with unity is called a division ring if every non-zero element is a unit.
- A commutative division ring is called a field.
- Every field is an I.D., but the converse is not true.
- Every finite I.D. is a field.

10.7 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 : The ring of integers Z in an I.D., but not a field.

Ans. to Q. No. 2 : Z_p = {0, 1, 2, ..., p–1}, p is a Prime. Clearly Z_p is a commutative ring with unity 1. It is finite as it has p elements.

Suppose $a \otimes b = 0$ for $a, b \in Z_p$.

Then $p \mid ab$

$\Rightarrow p \mid a$ or $p \mid b$, since p is prime

$\Rightarrow a = 0$ or $b = 0$, since $0 \le a, b < p$

Thus $a \otimes b = 0 \Rightarrow a = 0$ or $b = 0$, showing that Z_p has non-zero divisors. Hence Z_p is a finite integral domain and so, it is a field

Ans. to Q. No. 3 : Consider Z_8 = {0, 1, 2, 3, 4, 5, 6, 7}, the ring or integers modulo 8, where 8 is a composite number.

Here, $4 \otimes 6 = 0$ where as $4 \neq 0$, $6 \neq 0$.

Thus Z_8 has zero-divisors and so it cannot be a field.

Ans. to Q. No. 4 : See proof of Theorem 2.

10.8 FURTHER READINGS

1. *A course in Abstract Algebra*– V. K. Khanna & S. K. Bhambri.
2. *Modern Algebra*– S. Singh, Q. Zameeruddin.

10.9 MODEL QUESTIONS

Q.1. Find out the units of Z_8 = {0, 1, 2, 3, 4, 5, 6, 7}, the ring of integers modulo 8. Is it an I. D.? Is it a field?

Q.2. Show that M, the set of all 2 $\times$ 2 matrices of the type $\begin{pmatrix} Z_1 & Z_2 \\ -Z_2 & Z_1 \end{pmatrix}$ where Z_1, Z_2 are complex numbers is a division ring, but not a field.

Q.3. Show that R = {a + b$\sqrt{2}$: a, b $\in$ Q} is a field under usual addition and multiplication.

Q.4. Let R be the set of real numbers. Show that R $\times$ R forms a field under addition and multiplication defined by

(a, b) + (c, d) = (a + c, b + d)

(a, b).(c, d) = (ac – bd, ad + bc)

[Hints : additive identity is (0, 0), multiplicative identity is (1, 0)

If (a, b) $\neq$ (0, 0), then $(a, b)^{-1} = \left(\frac{a}{a^2+b^2}, -\frac{b}{a^2+b^2}\right)$]

Q.5. Write true of false :

i) Every field is a ring.

ii) Every ring has a multiplicative identity.

iii) Every ring with unity has atleast two units.

iv) Multiplication in a field is commutative.

v) The non-zero elements of a field form a multiplicative group.

vi) Every division ring is a field.

vii) Every integral domain is a field.

Q.6. Prove that Z_n is not a field, if n is a composite number.

UNIT 11 : MATRICES

UNIT STRUCTURE

11.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- define a matrix
- describe about types of matrix
- learn about the matrix operations i.e. addition, subtraction and multiplication
- define transpose of a matrix.

11.2 INTRODUCTION

Matrix is one of the most powerful tools in modern mathematics. They provide an algebraic structure slightly different from that of real numbers. The method of solving linear equations becomes easy with the help of matrices. Moreover, matrix notation and operations are used in computer graphics programming and implementation of electronic spreadsheet programs. They are widely used in modern algebra, applied mathematics, atomic physics, mathematical problems of economics as well as in computer science also. In this unit, we will introduce you to the fundamentals of matrix. We shall also discuss about the various types of matrices along with matrix operations.

11.3 DEFINITION OF MATRIX

A matrix is an ordered rectangular array of numbers.

A set of *'mn'* numbers (real or complex) arranged in the form of a rectangular array having *'m'* rows and *'n'* colums is called an $m \times n$ matrix (to be read *'m'* by *'n'* matrix).

An $m \times n$ matrix is usually written as :

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{bmatrix}$$

In a compact form the above matrix is represented by

$A = [a_{ij}]$, $i = 1, 2, \dots, \dots, m$

$j = 1, 2, \dots, \dots, n$

Or simply as $A = [a_{ij}]_{m \times n}$

We write the general elements of the matrix and enclose it in brackets of the type [ ] or ( ). The numbers $a_{11}, a_{12}, \dots,$ etc. of this rectangular array are called the elements or entries of the matrix. The element a_{ij} belongs to the i^{th} row and j^{th} column and is sometimes called the $(i, j)^{th}$ element of the matrix.

We denote matrices by capital letters such as A, B, C, etc. The following are some examples of matrices :

$A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ is a 2×2 matrix i.e. it has 2 rows and 2 column, whose elements are 2, 1, 3, 5.

$B = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 4 & -5 \end{bmatrix}$ is a 2×3 matrix. Whose elements are 3, –2, 0, 1, 4, –5.

$C = \begin{bmatrix} 1-i & 3 & 2/3 \\ 2.7 & -1 & 1 \\ \sqrt{2} & 5 & 4/7 \end{bmatrix}$ is a 3×3 matrix.

> **NOTE :** A matrix having *m*-rows and *n* column is called a matrix of order *m*×*n* or simply *m*×*n* matrix (read as *m* by *n* matrix). In the example, the matrix A is of order 2×2, B is of order 2×3 and C is of order 3×3.

11.4 TYPES OF MATRIX

Now, we are going to discuss about the different types of matrices.

i) Row Matrix : Any *1×n* matrix which has only one row and *n* columns is called a row matrix. e.g.

$A = [4 \;\; 6 \;\; -3 \;\; \sqrt{5} \;\; 0]$ is a row matrix of order 1×5.

In general, $A = [a_{1j}]_{1\times n}$ is a row matrix of order *1×n*.

ii) Column Matrix : Any *m×1* matrix which has *m* rows and only one column is called a column matrix. e.g.

$B = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ is a column matrix of order 3×1.

In general, $B = [b_{i1}]_{m\times 1}$ is a column matrix of order *m×1*.

iii) Square Matrix : A matrix in which the number of rows are equal to the number of columns is called a square matrix.

An *m*×*n* matrix for which *m* = *n* (i.e. the number of rows are equal to the number of columns) is called a square matrix of order *n*. For example,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1/2 & \sqrt{5} & 1 \\ 3 & 0 & -5 \end{bmatrix}$$ is a square matrix of order 3.

In a square matrix $A = [a_{ij}]_{n\times n}$, the elements $a_{11}, a_{22}, ..., a_{nn}$ are called the **Diagonal Elements**.

iv) Diagonal Matrix : A square matrix $B = [b_{ij}]_{m\times m}$ is said to be a diagonal matrix if all its non-diagonal elements are zero, that is a matrix $B = [b_{ij}]_{m\times m}$ is said to be a diagonal matrix if $b_{ij} = 0$, when $i \neq j$, e.g.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

are diagonal matrices of order 2, 3 respectively.

v) Scalar Matrix : A square matrix in which all the diagonal elements are equal and other elements are zero is called a scalar matrix. Thus, the square matrix $A = [a_{ij}]_{m\times m}$ is a scalar matrix if

$a_{ij} = 0$ when $i \neq j$ and $a_{11} = a_{22} = ... = a_{mm} = \alpha$ (say).

$$\text{e.g. } A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

are scalar matrices of order 2 and 3 respectively.

vi) Identity Matrix : A square matrix in which the diagonal elements are all 1 and rest are all zero is called an identity matrix.

Thus, a square matrix $A = [a_{ij}]_{m\times m}$ is an identity matrix if

$a_{ij} = 1$ when $i = j$

$a_{ij} = 0$ when $i \neq j$.

An $n\times n$ identity matrix is denoted by I_n. Thus

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are identity matrices of orders 2, 3 respectively. An identity matrix is also called a unit matrix. If there is no scope for confusion regarding the order of an identity matrix, then it is simply denoted by I.

vii) Null Matrix or Zero Matrix : A matrix of order $m \times n$ whose elements are all 0 is called a null matrix (or Zero matrix). It is usually denoted by 0.

$$\begin{bmatrix} 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

are all zero matrices of orders 1×2, 2×1, 2×2 respectively.

11.5 EQUALITY OF MATRICES

Two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ are said to be equal if :

i) they are of the same size or order.

ii) the elements in the corresponding places of the two matrices are equal i.e.,

$a_{ij} = b_{ij}$ for all i and j.

If two matrices A and B are equal, we write A = B. If two matrices A and B are not equal, we write A ≠ B. If two matrices are not of the same size, they cannot be equal, e.g.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

If $\begin{bmatrix} x & y \\ z & a \\ b & c \end{bmatrix} = \begin{bmatrix} -1.5 & 0 \\ 2 & \sqrt{6} \\ 3 & 2 \end{bmatrix}$

Then x = –1.5, y = 0, z = 2, a = $\sqrt{6}$, b = 3, c = 2

CHECK YOUR PROGRESS

Q.1. If a matrix has 12 elements, what are the possible orders it can have?

Q.2. Find the values of a, b, c and d from the following equation.

$$\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$$

Q.3. In a matrix $A = \begin{bmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & 5/2 & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{bmatrix}$

Find i) the order of the matrix,

ii) the number of elements,

iii) write the elements a_{13}, a_{21}, a_{33}, a_{24}, a_{23}.

11.6 ADDITION OF MATRICES

Suppose A and B be two matrices of the same order *m*×*n*, then their sum, denoted by A+B, is defined to be the matrix of the order *m*×*n* obtained by adding the corresponding elements of A and B.

Thus, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, then $A + B = [a_{ij} + b_{ij}]_{m\times n}$

The resultant matrix should be of the order of *m*×*n*.

More clearly, if $A = \begin{bmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{bmatrix}_{m\times n}$

and $B = \begin{bmatrix} b_{11} & b_{12} & \dots & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & \dots & b_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & \dots & b_{mn} \end{bmatrix}_{m\times n}$,

$$\text{then } A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & \dots & a_{2n}+b_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & \dots & a_{mn}+b_{mn} \end{bmatrix}_{m\times n}$$

Again, if, A and B are two matrices of order $m\times n$, then the difference 'A – B' is obtained as A – B = A + (–B), where –B is obtained on multiplying all the elements of B by –1.

It follows that $A - B = [a_{ij} - b_{ij}]_{m\times n}$.

whenever $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$.

Example : Given $A = \begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 6 \\ 1 & 7 \end{bmatrix}$,

find A + B and A – B.

Solution : $A + B = \begin{bmatrix} 1+3 & 5+6 \\ 0+1 & 3+7 \end{bmatrix} = \begin{bmatrix} 4 & 11 \\ 1 & 10 \end{bmatrix}$

$$A - B = A + (-B) = \begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix} + (-1)\begin{bmatrix} 3 & 6 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -3 & -6 \\ -1 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 1+(-3) & 5+(-6) \\ 0+(-1) & 3+(-7) \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & -4 \end{bmatrix}$$

In short, $A - B = \begin{bmatrix} 1-3 & 5-6 \\ 0-1 & 3-7 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & -4 \end{bmatrix}$

11.6.1 Properties of Matrix Addition

The addition of matrices satisfy the following properties :

i) Commutative Law : If $A = [a_{ij}]$, $B = [b_{ij}]$ are matrices of the same order, say $m\times n$, then A + B = B + A

Now, $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

$= [b_{ij} + a_{ij}]$ (addition of numbers is commutative)

$= [b_{ij}] + [a_{ij}]$

$= B + A$

ii) Associative Law : For any three matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$ of the same order, say $m \times n$,

$(A + B) + C = A + (B + C)$

$$\begin{aligned} \text{Now, } (A + B) + C &= ([a_{ij}] + [b_{ij}]) + [c_{ij}] \\ &= [a_{ij} + b_{ij}] + [c_{ij}] \\ &= [(a_{ij} + b_{ij}) + c_{ij}] \\ &= [a_{ij} + (b_{ij} + c_{ij})] \\ &= [a_{ij}] + ([b_{ij}] + [c_{ij}]) \\ &= A + (B + C) \end{aligned}$$

iii) Existence of Additive Identity : If *0* be the $m \times n$ zero matrix and $A = [a_{ij}]$ be an $m \times n$ matrix, then

$A + 0 = [a_{ij} + 0]_{m\times n} = [a_{ij}]_{m\times n} = A$

also, $0 + A = [0 + a_{ij}]_{m\times n} = [a_{ij}]_{m\times n} = A$

In other words, the null matrix plays the role of additive identity in the set of all $m \times n$ matrices.

iv) Existence of Additive Inverse : Let $A = [a_{ij}]_{m\times n}$ be any matrix. Then we have another matrix $-A = [-a_{ij}]_{m\times n}$, such that

$A + (-A) = (-A) + A = 0$

So, –A is the additive inverse of A or negative of A.

Example : If $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ and $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ find A + B.

Solution : $A + B = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$$= \begin{bmatrix} a+a & b-b \\ -b+b & a+a \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 0 & 2a \end{bmatrix}$$

Example : If $A = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 3 \\ 2 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 5 \\ 6 & 4 \end{bmatrix}$,

Show that (A + B) + C = A + (B + C).

Solution : $A + B = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+3 \\ 3+2 & 4+5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 9 \end{bmatrix}$

$$\therefore (A + B) + C = \begin{bmatrix} 1 & 3 \\ 5 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} 1+3 & 3+5 \\ 5+6 & 3+4 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 11 & 13 \end{bmatrix}$$

$$\text{Again, } B + C = \begin{bmatrix} 0 & 3 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} 0+3 & 3+5 \\ 2+6 & 5+4 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 9 \end{bmatrix}$$

$$\therefore A + (B + C) = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 8 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 1+3 & 0+8 \\ 3+8 & 4+9 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 11 & 13 \end{bmatrix}$$

$\therefore$ (A + B) + C = A + (B + C).

11.7 MULTIPLICATION OF A MATRIX BY A SCALAR

Multiplication of a matrix by a scalar is defined as follows :

If $A = [a_{ij}]_{m\times n}$ is a matrix of order $m\times n$ and k is a scalar, then kA is an another matrix which is obtained by multiplying each element of A by the scalar k.

In other words, $kA = k[a_{ij}]_{m\times n} = [k.a_{ij}]_{m\times n}$

i.e. $(i, j)^{th}$ element of kA is ka_{ij} for all possible values of i and j.

$$\text{For example, if } A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}$$

$$\text{then} \quad 2A = 2\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -2 \\ 4 & 0 & 6 \\ 6 & -2 & 4 \end{bmatrix}$$

$$\frac{1}{2}A = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & \frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

Conversely, if all the elements of a matrix have a common factor, then that common factor can be taken out. For example,

$$\begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix} = \alpha\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

11.7.1 Properties of Multiplication of a Matrix by a Scalar

If A = $[a_{ij}]$ and B = $[b_{ij}]$ are two matrices of same order, say $m \times n$ and k and l are two scalars, then the following are true.

i) k(A + B) = kA + kB

ii) (k + l)A = kA + lA

iii) k(lA) = (kl)A

iv) (–k)A = k(–A)

Example : If $Y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$, then find the matrix X.

Solution : We have $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$

$$\Rightarrow 2X = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} - Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3 & 0-2 \\ -3-1 & 2-4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix}$$

$$\Rightarrow X = \frac{1}{2}\begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix} \qquad \therefore X = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$$

Example : If $A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 6 & -3 \\ 1 & 0 & -2 \end{bmatrix}$, then find the matrix X such that 2A – 3B – X = 0

Solution : We have 2A – 3B – X = 0

$$\therefore X = 2A - 3B$$

$$= 2\begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} - 3\begin{bmatrix} 4 & 6 & -3 \\ 1 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 & 6 \\ 4 & 2 & 8 \end{bmatrix} - \begin{bmatrix} 12 & 18 & -9 \\ 3 & 0 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 0-12 & 4-18 & 6-(-9) \\ 4-3 & 2-0 & 8-(-6) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & -14 & 15 \\ 1 & 2 & 14 \end{bmatrix}$$

11.8 MULTIPLICATION OF MATRICES

Two matrices A and B are conformable for the product AB only when the number of columns of A is equal to the number of rows of B.

Let A = $[a_{ij}]_{m\times n}$ i.e. A is of order $m\times n$ and B = $[b_{jk}]_{n\times p}$ i.e. B is of order $n\times p$. Then the product AB of the matrices A and B is the matrix C of order $m\times p$, i.e.

C = $[c_{ik}]_{m\times p}$

Where, $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + a_{i3}b_{3k} + \ldots \ldots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}.b_{jk}$

It means, the $(i, k)^{th}$ element c_{ik} of the matrix C = AB is obtained by multiplying the corresponding elements of the i^{th} row of A and the k^{th} column of B and then adding the products. The rule of multiplication is row by column multiplication i.e. in the process of multiplication we take the rows of A and the columns of B.

In the product AB the matrix A is called the *prefactor* and the matrix B is called the *post factor*.

NOTE :
1) The product AB of two matrices A and B exists if and only if the number of columns in A is equal to the number of rows in B.
2) If A be of order m×n and B be of order n×p, then C = AB is defined and will be of order m×p.

Example : If A = $\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 0 & 4 \end{bmatrix}$ find AB. Also find BA. Are they equal?

Solution : Here, the matrix A is of order 3×1 and B is of order 1×3. Since the number of columns in A is equal to the number of rows in B, so the two matrix will be comformable for the product AB. The resultant matrix will be of order 3×3.

$$AB = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 3\times1 & 3\times0 & 3\times4 \\ 5\times1 & 5\times0 & 5\times4 \\ 2\times1 & 2\times0 & 2\times4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 12 \\ 5 & 0 & 20 \\ 2 & 0 & 8 \end{bmatrix}$$

Again, B is 1×3 and A is 3×1. Here, column number in B is equal to the row number in A. The resultant matrix will be of order 1×1.

$$BA = \begin{bmatrix} 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = [1\times3 + 0\times5 + 4\times2] = [11]$$

$\therefore$ AB $\neq$ BA.

NOTE : Matrix multiplication is not commutative i.e. for two matrices A and B which are comformable for multiplication, AB $\neq$ BA.

Example : If $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 & 2 \\ 1 & -2 & 3 \end{bmatrix}$ find AB.

Can you find BA?

Solution : Here, matrix A is of order 2×2 and matrix B is of order 2×3. Number of columns in matrix A is equal to the number of rows in B, so the product AB is defined and the resultant matrix will be of order 2×3.

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 3+1\times 1 & 2\times 0+1\times(-2) & 2\times 2+1\times 3 \\ 1\times 3+0\times 1 & 1\times 0+0\times(-2) & 1\times 2+0\times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6+1 & 0-2 & 4+3 \\ 3+0 & 0 & 2+0 \end{bmatrix} = \begin{bmatrix} 7 & -2 & 7 \\ 3 & 0 & 2 \end{bmatrix}$$

Since number of columns of B is not equal to the number of rows of A, the product BA is undefined.

11.8.1 Properties of Multiplication of Matrix

The multiplication of matrices possesses the following properties :

i) The multiplication of matrices is not always commulative :

a) Whenever AB exists, it is not always necessary that BA should also exist.

b) Whenever AB and BA both exist, it is always not necessary that they should be matrices of the same order.

c) Whenever AB and BA both exist and are matrices of the same order, it is not necessary that AB = BA.

For example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

then we can show that AB ≠ BA although both AB, BA exist.

d) It however does not imply that AB is never equal to BA.

For example, if $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 3 & 2 \end{bmatrix}$ and

$B = \begin{bmatrix} 10 & -4 & -1 \\ -11 & 5 & 0 \\ 9 & -5 & 1 \end{bmatrix}$, we can show that AB = BA.

ii) Matrix multiplication is associative i.e. A(BC) = (AB)C where A, B and C are matrices of order $m \times n$, $n \times p$ and $p \times q$ respectively.

iii) Matrix multiplication is distributive with respect to addition of matrices i.e. A(B + C) = AB + AC
where A, B and C are matrices of order $m \times n$, $n \times p$ and $n \times p$ respectively.

iv) For every square matrix A, there exist an identity matrix *I* of same order such that, AI = IA = A.

CHECK YOUR PROGRESS

Q.4. Find X and Y if $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

Q.5. Find the values of x and y from the following equation :

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Q.6. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$.

Q.7. Using examples, show that A(BC) = (AB)C.

Q.8. Using examples, show that A(B+C) = AB + AC.

11.9 TRANSPOSE OF A MATRIX

Let $A = [a_{ij}]_{m \times n}$ be a matrix of order $m \times n$. Then the matrix obtained from A by changing its rows into columns and columns into rows is called the transpose of A and is denoted by the symbol A' or A^T.

i.e. if $A = [a_{ij}]_{m \times n}$ then $A' = [b_{ij}]_{n \times m}$, where $b_{ij} = a_{ji}$

i.e. the $(i, j)^{th}$ element of A' is the $(j, i)^{th}$ element of A.

For example : If $A = \begin{bmatrix} 1 & 9 & 7 \\ 4 & 3 & 5 \end{bmatrix}$, then $A' = \begin{bmatrix} 1 & 4 \\ 9 & 3 \\ 7 & 5 \end{bmatrix}$

Some important properties of transpose of matrices are given below :

1. The transpose of the transpose of a matrix is the matrix itself i.e. if A is a matrix of order $m \times n$, then $(A')' = A$.
2. If A is matrix of order $m \times n$ and k is a scalar, then $(kA)' = kA'$

3. If A and B are two matrices suitable for addition, then

   $(A+B)' = A' + B'$

4. If A and B are two matrices such that AB is defined, then

   $(AB)' = B'A'$

Example 1 : If $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$ show that $(A')' = A$.

Solution : We have $A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$ $\therefore A' = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$

Now $(A')' = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix} = A$

Hence $(A')' = A$

Example 2 : If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 5 & 1 \\ 6 & 7 & -1 \end{pmatrix}$,

then show that $(AB)' = B'A'$.

Solution : $A' = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, $B' = \begin{pmatrix} 4 & 6 \\ 5 & 7 \\ 1 & -1 \end{pmatrix}$,

and so $B'A' = \begin{pmatrix} 4 & 6 \\ 5 & 7 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 16 & 36 \\ 19 & 43 \\ -1 & -1 \end{pmatrix}$

Also $AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 4 & 5 & 1 \\ 6 & 7 & -1 \end{pmatrix} = \begin{pmatrix} 16 & 19 & -1 \\ 36 & 43 & -1 \end{pmatrix}$

$\therefore (AB)' = \begin{pmatrix} 16 & 36 \\ 19 & 43 \\ -1 & -1 \end{pmatrix}$ Thus $(AB)' = B'A'$.

11.10 SYMMETRIC AND SKEW-SYMMETRIC MATRICES

1. **Symmetric Matrix :** A square matrix $A = [a_{ij}]$ is said to be symmetric if $A' = A$ i.e. $[a_{ij}] = [a_{ji}]$ for all possible values of *i* and *j*, its $(i, j)^{th}$ element is the same as its $(j, i)^{th}$ element.

The matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 7 \\ 3 & 7 & 6 \end{bmatrix}$ is a symmetric

For A′ = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 7 \\ 3 & 7 & 6 \end{bmatrix}$ = A

Examples : $\begin{bmatrix} 2 & 7 \\ 7 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & i & -2i \\ i & -2 & 4 \\ -2i & 4 & 3 \end{bmatrix}$ are symmetric matrices.

NOTE :

i) A square matrix A is said to be symmmetric if A = A′

ii) A square matrix A is said to be skew-symmetric if A′ = –A

iii) The diagonal elements of a skew-symmetric matrix are all zero.

2. **Skew-Symmetric Matrix :** A square matrix A = $[a_{ij}]$ is said to be skew-symmetric if A′ = –A that is $a_{ij} = -a_{ji}$ for all possible values of *i*, *j*. In particular, for *i* = *j*, we have $a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$

Thus, the diagonal elements of a skew-symmetric matrix are all zero.

A = $\begin{bmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{bmatrix}$ is skew-symmetric

for A′ = $\begin{bmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{bmatrix} = -\begin{bmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{bmatrix}$ = –A

Examples : $\begin{bmatrix} 0 & -3i & -4 \\ 3i & 0 & 8 \\ 4 & 8 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ are skew-symmetric matrices.

11.11 CANONICAL FORM OF MATRICES

Definition 1 :

ECHELON MATRIX : A matrix A is called an **echelon matrix**, or is said to be in **echelon form** if :

i) any zero rows, that is rows with all elements zero, are on the bottom of the matrix.

ii) the first non-zero entry, called the **leading entry** in a row, occurs to the right of the leading non-zero entry in the preceding row.

Example : The matrices $\begin{pmatrix} 1 & 2 & -3 \\ 0 & 4 & -20 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & 4 & 5 \end{pmatrix}$

are echelon matrices.

Definition 2 :

ROW CANONICAL FORM OF MATRIX : A matrix A is said to be in **row canonical form** if :

i) A is an echelon matrix,

ii) each leading non-zero entry in a row is 1,

iii) each leading non-zero entry is the only non-zero entry in its column.

A matrix in Row Canonical Form is also called **Reduced Echelon Matrix**.

Example : The matrix $\begin{pmatrix} 1 & 0 & 5 & 0 & 2 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ is in row canonical form

where as the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is not in row canonical form.

Any given matrix can be reduced to row canonical form with the help of some operations performed in its rows. These are called **Elementary Row Operations** which we state below.

ELEMENTARY ROW OPERATIONS : The following are called elementary row operations or a matrix $A = [a_{ij}]_{m\times n}$ with respective symbols :

i) Interchanging the i^{th} row and the j^{th} row : $R_i \leftrightarrow R_j$.

ii) Multiplying the i^{th} row by a non-zero scalar k : $R_i \to kR_i$.

iii) Replacing the i^{th} row by adding to this row k times the j^{th} row :

$R_i \to R_i + kR_j$.

EQUIVALENT MATRICES : If A and B are two m×n matrices, then B is said to be **row-equivalent** to A if it can be obtained from A by a finite number of elemantary row operations. This row-equivalence is then denoted by $A \xrightarrow{R} B$ or by A ~ B.

It can be shown that $A \sim B \Leftrightarrow B \sim A$

$A \sim B,\ B \sim C \Rightarrow A \sim C$.

Example 1 : Using elementary row operations, show that

$$B = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 2 \\ 0 & 3 & 0 & 5 \end{pmatrix} \text{ is row equivalent to } A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ -1 & 1 & 0 & 2 \end{pmatrix}.$$

Solution : $A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ -1 & 1 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 3 & 0 & 5 \\ -1 & 1 & 0 & 2 \end{pmatrix}$,

by $R_2 \to R_2 + 2R_3$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 2 \\ 0 & 3 & 0 & 5 \end{pmatrix} = B, \text{ by } R_2 \leftrightarrow R_3$$

Thus A ~ B.

Example 2 : Reduce the matrix $A = \begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix}$

to echelon form.

Solution : $A = \begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 7 & -7 & 6 \end{pmatrix}, \text{ by } R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$$

$$\sim \begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 0 & 7 & -10 \end{pmatrix}, \text{ by } R_3 \to 3R_3 - 7R_2$$

Thus A $\sim \begin{pmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 0 & 7 & -10 \end{pmatrix}$ which is in echelon form.

Example 3 : Reduce A = $\begin{pmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \end{pmatrix}$ to echelon form.

Solution : A $= \begin{pmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \end{pmatrix}$

$\sim \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 2 & 3 & 2 & -1 \end{pmatrix}$, by $R_1 \leftrightarrow R_2$

$\sim \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & -4 \end{pmatrix}$, by $R_3 \rightarrow R_3 - R_1$

$\sim \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, by $r_3 \rightarrow R_3 - 2R_2$

Thus A $\sim \begin{pmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ which is in echelon form.

Example 4 : Reduce the matrix A = $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 7 & 4 \\ 4 & 7 & 10 & 13 & 16 \end{bmatrix}$

to row canonical form.

Solution : A $= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 7 & 4 \\ 4 & 7 & 10 & 13 & 16 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & -1 & -3 & -5 & -11 \\ 0 & -1 & -2 & -3 & -4 \end{bmatrix}, \text{ by } R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - 3R_1, R_4 \to R_4 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -5 & -11 \\ 0 & -1 & -2 & -3 & -4 \end{bmatrix}, \text{ by } R_2 \to (-1).R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & -2 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ by } R_3 \to R_3 + R_2, R_4 \to R_4 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ by } R_3 \to (-1).R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ by } R_2 \to R_2 - 2R_3$$

Which is in row canonical form.

Example 5 : Reduce to row canonical form :

$$A = \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 1 & 1 & 4 & -1 & 3 \\ 2 & 5 & 9 & -2 & 8 \end{bmatrix}$$

Solution : $A = \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 1 & 1 & 4 & -1 & 3 \\ 2 & 5 & 9 & -2 & 8 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 9 & 3 & -4 & 4 \end{bmatrix}, \text{ by } R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}, \text{ by } R_3 \to R_3 - 3R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}, \text{ by } R_3 \to \tfrac{1}{2}R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 0 & \frac{3}{2} \\ 0 & 3 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}, \text{ by } R_1 \to R_1 - R_3,\ R_2 \to R_2 + 2R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}, \text{ by } R_2 \to \frac{1}{3}R_2$$

$$\sim \begin{bmatrix} 1 & -2 & \frac{11}{3} & 0 & \frac{17}{6} \\ 0 & 1 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}, \text{ by } R_1 \to R_1 + 2R_2$$

Which is the row canonical form of A.

11.12 LET US SUM UP

- A rectangular array of *mn* numbers, arranged in *m*-rows and *n*-columns and enclosed in a square [ ] or a round bracket ( ), is called a *matrix* of order *m* by *n* (denoted by *m*×*n*).

- The numbers occuring in a matrix are called the elements of a matrix.
- The element a_{ij} appearing in the i^{th} row and the j^{th} column of A is called the $(i, j)^{th}$ element of A.
- A matrix with exactly one column and any number of rows is called a column *matrix*. The order of a column matrix is of the type $m \times 1$.
- A matrix with exactly one row and any number of columns is called a row matrix. The order of a row matrix is of the type $1 \times n$.
- A matrix $A = [a_{ij}]_{m\times n}$ is a square matrix if $m = n$.
- The square matrix $[a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$ for $i \neq j$.
- The square matrix $[a_{ij}]$ is a scalar matrix if $a_{ij} = \begin{cases} \alpha & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}$
- The square matrix $[a_{ij}]$ is a unit matrix if $a_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}$

  It is denoted by I_n or I.
- A matrix in which each element is zero is called a zero matrix or a null matrix, denoted by 0.
- Two matrix are said to be equal if and only if they are of the same order and their corresponding elements are equal.
- If A and B are two matrices of the same order, then their sum, denoted by A + B, is the matrix obtained by adding the corresponding elements of A and B.
- Commutative property of addition. For two matrix A and B, A+B = B+A.
- Associativity of addition : (A+B) + C = A + (B+C)
- Existence of additive identity : A + 0 = 0 + A = A
- Existence of additive inverse : A + (–A) = (–A) + A = 0
- Scalar multiplication of matrix : $A = [a_{ij}]_{m\times n} \Rightarrow kA = [k.a_{ij}]_{m\times n}$

  a) k(A + B) = kA + kB

  b) $(k+l)A = kA + lA$

  c) $k(lA) = (kl)A$

  d) (–k)A = k(–A)
- If $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$, then $AB = C = [c_{ik}]_{m\times p}$,

where $c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}$

- Matrix multiplication is not commutative in general, i.e. $AB \neq BA$.
- Matrix multiplication is associative, i.e. $(AB)C = A(BC)$.
- Matrix multiplication is distributive over addition,
  i) $A(B+C) = AB + AC$
  ii) $(A+B)C = AC + BC$
- If A is a square matrix and *I* is an identity matrix having the order same as A then $AI = IA = A$.
- If $A = [a_{ij}]_{m\times n}$, then A' or $A^T = [b_{ji}]_{n\times m}$ where $b_{ji} = a_{ij}$, is called the transpose of A.
- Matrix A is called a symmetric matrix if $A' = A$.
- A is skew symmetric matrix if $A' = -A$.
- A matrix A is in echelon form if :
  i) any zero rows are on the bottom of the matrix.
  ii) The first non-zero entry, called the leading entry in a row occurs to the right of the leading non-zero entry in the preceding row.
- A matrix A is said to be in row canonical form if :
  i) A is an echelon matrix,
  ii) each leading non-zero entry in a row is 1,
  iii) each leading non-zero entry is the only non-zero entry in its column.
- Elementary row operations are :
  i) interchanging any two rows : $R_i \leftrightarrow R_j$
  ii) multiplying a row by a non-zero scalar : $R_i \rightarrow kR_j$
  iii) replacing the i^{th} row by adding to this row k times the j^{th} row : $R_i \rightarrow R_i + kR_j$.
- Two matrices A and B are row-equivalent, denoted by $A \sim B$, if B can be obtained from A using finite sequence of elementary row operations.

11.13 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 : We have already discussed that if a matrix is of order $m \times n$, it has *'mn'* elements. Now to find all possible orders of a matrix with 12 elements, we will find all ordered pairs of natural numbers, whose product is 12. Thus, all possible ordered pairs are :

(1, 12), (12, 1), (2, 6), (6, 2), (3, 4), (4, 3)

Hence, possible orders are :

1×12, 12×1, 2×6, 6×2, 3×4, 4×3.

Ans. to Q. No. 2 : By equality of two matrices, equating the corresponding elements we get $2a + b = 4$ $\quad a - 2b = -3$

$5c - d = 11$ $\quad 4c + 3d = 24$

Solving these equations we will get a = 1, b = 2, c = 3, d = 4

Ans. to Q. No. 3 : i) 3×4; ii) 12; iii) 19, 35, –5, 12, $\frac{5}{2}$

Ans. to Q. No. 4 : We have $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$, $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

Adding the two, we get

$$X + Y + X - Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\Rightarrow 2X = \begin{bmatrix} 7+3 & 0+0 \\ 2+0 & 4+3 \end{bmatrix}$$

$$\Rightarrow X = \frac{1}{2}\begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

$$\text{Now, } X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$$

$$\Rightarrow Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} - X$$

$$= \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$ [Putting the value of X]

$$= \begin{bmatrix} 7-5 & 0-0 \\ 2-1 & 5-4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}, Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Ans. to Q. No. 5 : We have, $2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x+3 & 10-4 \\ 14+1 & 2y-6+2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$\Rightarrow 2x + 3 = 7$ and $2y - 4 = 14$

$\Rightarrow 2x = 7 - 3$ and $2y = 14 + 4$

$\Rightarrow x = \frac{4}{2}$ and $y = \frac{18}{2}$

$\therefore x = 2$ and $y = 9$

Ans. to Q. No. 6 : We have $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$\text{Now } A^2 = A.A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$5A = 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$$

$$7I = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$\therefore A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} -15 & -5 \\ 5 & -10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence, $A^2 - 5A + 7I = 0$.

Ans. to Q. No. 7 : Take $A = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix}$

Clearly AB, BC are defined and we get.

$$AB = \begin{pmatrix} 2 & 1 & 5 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 15 & 15 \\ 4 & 12 \end{pmatrix}$$

$$BC = \begin{pmatrix} 3 & 4 \\ -1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ -3 & -5 \\ 1 & 5 \end{pmatrix}$$

Now, $A(BC) = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} -1 & 5 \\ -3 & -5 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 30 \\ -8 & 0 \end{bmatrix}$

$$(AB)C = \begin{bmatrix} 15 & 15 \\ 4 & 12 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 30 \\ -8 & 0 \end{bmatrix}$$

Thus A(BC) = (AB)C.

Ans. to Q. No. 8 : Take $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$

Clearly AB, AC, B+C and A(B+C) are defined.

$$B+C = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$$

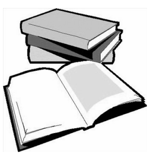
$$\therefore A(B+C) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 1 & -1 \\ 5 & 3 \end{bmatrix}$$

$$\text{Again, } AB = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -1 \\ 6 & -1 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 0 \\ -1 & 4 \end{bmatrix}$$

$$\therefore AB + AC = \begin{bmatrix} -2 & 2 \\ 2 & -1 \\ 6 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ -1 & 0 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 1 & -1 \\ 5 & 3 \end{bmatrix}$$

Thus A(B+C) = AB + AC.



11.14 FURTHER READINGS

1. *Matrix Algebra*, S. K. Jain, Eastern Book House.
2. *Matrices*, A. R. Vasishtha, Krishna Prakashan Mandir, Meerut.

11.15 MODEL QUESTIONS

Q.1. If $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$ find A – 3B.

Q.2. If $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ 4 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$.

Verify the following : i) A + B = B + A ii) 2(A – B) = 2A – 2B

Q.3. Solve for x, $\begin{bmatrix} x^2 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2x & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 3 & 7 \end{bmatrix}$

Q.4. Find x and y if $x-2y = \begin{bmatrix} -1 & 4 \\ -3 & 1 \end{bmatrix}$ and $2x-y = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$

Q.5. If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 4 \\ 2 & 1 \end{bmatrix}$

find AB and BA and show that AB ≠ BA.

Q.6. If A = $\begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}$ find $-A^2 + 6A$.

Q.7. If A = $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, show that $A^2 - 2A = 3I$.

Q.8. If A = $\begin{pmatrix} 2 & -3 & 5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix}$, show that $A^2 = A$.

Q.9. If A = $\begin{pmatrix} 6 & 9 \\ -4 & -6 \end{pmatrix}$, show that $A^2 = 0$.

Q.10. If n is any positive integer, show that

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}^n = \begin{bmatrix} \cos nx & \sin nx \\ -\sin nx & \cos nx \end{bmatrix}$$

Q.11. Give examples to show that, $(A+B)' = A' + B'$, $(AB)' = B'A'$.

Q.12. If A be a square matrix, show that :

i) AA' is symmetric matrix.

ii) $A + A'$ is symmetric and $A - A'$ is skew-symmetric.

iii) A is the sum of a symmetric and a skew-symmetric matrix.

Q.13. Reduce to echelon form :

i) $\begin{pmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -4 & 3 \end{pmatrix}$

ii) $\begin{pmatrix} 2 & -2 & 2 & 1 \\ -3 & 6 & 0 & -1 \\ 1 & -7 & 10 & 2 \end{pmatrix}$

Q.14. Reduce to row-canonical form :

i) $\begin{pmatrix} 1 & 2 & -3 & 0 & 1 \\ 0 & 0 & 5 & 2 & -4 \\ 0 & 0 & 0 & 7 & 3 \end{pmatrix}$

ii) $\begin{pmatrix} 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ 6 & 6 & 0 & 20 & 19 \end{pmatrix}$

UNIT 12 : DETERMINANT - I

UNIT STRUCTURE

12.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- know about determinants of a square matrix, minors and cofactors of elements of a matrix
- define adjoint of a square matrix and know how to find it
- define inverse of a square matrix
- find inverse of a square matrix provided it exists
- define rank and nullity of a matrix
- find rank using definition and using elementary row operations.

12.2 INTRODUCTION

One of the important topics in Algebra is that of solution of a system of linear equations. The system is expressed in matrix form as a single equation in matrices. To determine the existence of solution of the system, the concept of rank of matrix is necessary. Also the concept of inverse of a matrix is used to find the solution set. In this unit, we shall learn about the inverse of a matrix and rank and nullity of a matrix. For this purpose the concept of determinant of a square matrix and that of adjoint of a square matrix is necessary which will be discussed at the beginning of the unit.

12.3 DETERMINANT OF A SQUARE MATRIX

1) Let A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a 2×2 square matrix.

We define its determinant, denoted by **detA** or |**A**|, as

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

It is called a determinant of **order 2**.

For example, if A = $\begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}$, than |A| = $\begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix}$ = 1.3 – 2.(–2) = 7

2) If A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a 3×3 square matrix, its determinant,

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$ is given by

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

NOTE : Let F be a field and $M_n(F)$ be the set of all square matrices of order n with entries from the field F. Then $M_n(F)$ is a non-commutative ring. We can define a mapping from $M_n(F)$ to F such that the image of $A = (a_{ij})_{n\times n} \in M_n(F)$ is the determinant of A, i.e., $|A| \in F$.

$M_n(F) \to F$

$A \to |A|$

Thus, the determinant of a matrix can be considered as a function.

If $A = (a_{11})_{1\times 1}$ then $|A| = |a_{11}| = a_{11}$

In general, $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$= (-1)^{i+1}a_{i1}|A_{i1}| + (-1)^{i+2}a_{i2}|A_{i2}| + (-1)^{i+3}a_{i3}|A_{i3}|$

Where A_{ij} represents the 2×2 matrix obtained on deleting the i^{th} row and j^{ih} column of A(i, j = 1, 2, 3).

Similarly, $|A| = (-1)^{1+j}a_{1j}|A_{1j}| + (-1)^{2+j}a_{2j}|A_{2j}| + (-1)^{3+j}a_{3j}|A_{3j}|$

This determinant is called a determinant of **order 3**.

For example, if $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ -2 & 1 & 4 \end{pmatrix}$, then

$$|A| = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ -2 & 1 & 4 \end{vmatrix}$$

$$= 1.\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} - 2.\begin{vmatrix} 2 & 3 \\ -2 & 4 \end{vmatrix} + (-1).\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix}$$

$= 1.(0-3) - 2(8+6) - 1.(2-0)$

$= -3 - 28 - 2 = -33$

Similarly, we can define the determinant of a square matrix of order 4, 5, ..., n.

12.3.1 Minors and Cofactors

If $A = [a_{ij}]_{n\times n}$ be a square matrix, then the determinant of the $(n-1)\times(n-1)$ matrix A_{ij} obtained on deleting the i^{th} row and j^{th} column of A is called the **minor of a_{ij}**, i.e., minor of $a_{ij} = |A_{ij}|$,

where A_{ij} is the $(n-1)\times(n-1)$ matrix as defined above.

The **cofactor of a_{ij}**, denoted by C_{ij}, is defined as

$C_{ij} = (-1)^{i+j} \times$ minor of $a_{ij} = (-1)^{i+j}|A_{ij}|$.

Example 1 : Find minors & cofactors of the elements of

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Solution : The minor of a_{11} is $|A_{11}| = |a_{22}| = a_{22}$

The minor of a_{12} is $|A_{12}| = |a_{21}| = a_{21}$

The minor of a_{21} is $|A_{21}| = |a_{12}| = a_{12}$

The minor of a_{22} is $|A_{22}| = |a_{11}| = a_{11}$

So, the cofactors are

$C_{11} = (-1)^{1+1}|A_{11}| = a_{22}$

$C_{12} = (-1)^{1+2}|A_{12}| = -a_{21}$

$C_{21} = (-1)^{2+1}|A_{21}| = -a_{12}$

$C_{22} = (-1)^{2+2}|A_{22}| = a_{11}$

Clearly $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$= a_{11}a_{22} - a_{12}a_{21}$

$= a_{11}C_{11} + a_{12}C_{12}$.

Similarly, $|A| = a_{21}C_{21} + a_{22}C_{22}$

$= a_{11}C_{11} + a_{21}C_{21}$

$= a_{12}C_{12} + a_{22}C_{22}$

Also $a_{11}C_{21} + a_{12}C_{22} = a_{11}(-a_{12}) + a_{12}.a_{11} = 0$, etc.

Example 2 : Find the cofactors of the elements of

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Solution : $C = (-1)^{1+1}|A_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$$C_{12} = (-1)^{1+2}|A_{12}| = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{13} = (-1)^{1+3}|A_{13}| = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \text{ etc.}$$

Clearly, $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$, etc.

Also, we can show that $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$, etc.

We can summerize the findings in examples 2 and 3 in the following property.

Property : 1) The sum of the products of the elements of any row (column) and the corresponding cofactors of the row (column) is equal to the value of the determinant.

2) The sum of the products of the elements of any row (column) and the cofactors of the corresponding elements of a different row (column) is equal to zero.

12.3.2 Determinant of a Diagonal Matrix

Let $A = \begin{bmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_n \end{bmatrix}$ be a diagonal matrix of order n. Then

$|A| = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$

$= d_1C_{11}$, as $a_{12} = \dots = a_{1n} = 0$

$$= d_1 \begin{bmatrix} d_2 & 0 & 0 & \dots & 0 \\ 0 & d_3 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_n \end{bmatrix}$$

$$= d_1 d_2 \begin{bmatrix} d_3 & 0 & 0 & \dots & 0 \\ 0 & d_4 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_n \end{bmatrix}$$

$= \dots \dots \dots \dots \dots$

$= d_1\, d_2\, d_3 \dots d_n$

As a corollary, we get $|I_n| = 1$.

12.3.3 Determinant of Product of Two Matrices

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ then $|A| = ad - bc$, $|B| = \alpha\delta - \beta\gamma$

$\therefore$ |A| |B| = $(ad-bc)(\alpha\delta-\beta\gamma)$

$= ad\alpha\delta + bc\beta\gamma - ad\beta\gamma - bc\alpha\delta$ ....(1)

Also AB = $\begin{bmatrix} a\alpha+b\gamma & a\beta+b\delta \\ c\alpha+d\gamma & c\beta+d\delta \end{bmatrix}$

$\therefore$ |AB| = $\begin{vmatrix} a\alpha+b\gamma & a\beta+b\delta \\ c\alpha+d\gamma & c\beta+d\delta \end{vmatrix}$

$= (a\alpha+b\gamma)(c\beta+d\delta) - (a\beta+b\delta)(c\alpha+d\gamma)$

$= ad\alpha\delta + bc\beta\gamma - ad\beta\gamma - bc\alpha\delta$ ....(2)

From (1) & (2), we get |AB| = |A|.|B|

In general, if $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$ be two square matrices of same order, then |AB| = |A|.|B|.

12.4 ADJOINT OF A SQUARE MATRIX

Definition 1 : Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order n. Then the adjoint of A, denoted by adjA, is the transpose of the matrix of the cofactors of the corresponding elements of A.

Thus, if A = $\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix}$

then adjA = $\begin{bmatrix} c_{11} & c_{12} & ... & c_{1n} \\ c_{21} & c_{22} & ... & c_{2n} \\ ... & ... & ... & ... \\ c_{n1} & c_{n2} & ... & c_{nn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{21} & ... & c_{n1} \\ c_{12} & c_{22} & ... & c_{n2} \\ ... & ... & ... & ... \\ c_{1n} & c_{2n} & ... & c_{nn} \end{bmatrix}$

Example 3 : Let A = $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{bmatrix}$

Then, $C_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -1 \\ 2 & 5 \end{vmatrix} = 7$ $\quad C_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} = -11$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ $\quad C_{21} = (-1)^{2+1}\begin{vmatrix} 0 & -1 \\ 2 & 5 \end{vmatrix} = -2$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -1 \\ 1 & 5 \end{vmatrix} = 6 \qquad C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = -2$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} = 1 \qquad C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -1$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$$

$$\therefore \text{ adjA} = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} = \begin{bmatrix} 7 & -2 & 1 \\ -11 & 6 & -1 \\ 3 & -2 & 1 \end{bmatrix}$$

Theorem 1 : If A be an n×n square matrix, then

$A(\text{adj}A) = |A|\, I_n = (\text{adj}A)A$.

Proof : Let $A = (a_{ij})_{n\times n}$.

Then $\text{adj}A = (b_{ij})_{n\times n}$, where $b_{ij} = C_{ji}$, the cofactor of a_{ji}.

Now, the (i, j)th element of A(adjA) $= \sum_{k=1}^{n} a_{ik} b_{kj}$

$$= \sum_{k=1}^{n} a_{ik} C_{jk}$$

$$= \begin{cases} |A|, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

$$\text{Hence, } A(\text{adj}A) = \begin{bmatrix} |A| & 0 & 0 & \dots & 0 \\ 0 & |A| & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$= |A|.I_n$$

Similarly, $(\text{adj}A) = |A|.I_n$

Thus, $A(\text{adj}A) = |A|\, I_n = (\text{adj}A)A$.

12.5 INVERSE OF A SQUARE MATRIX

Definition 2 : Let A be a square matrix of order n. If there exists another square matrix of order n such that $AB = I_n = BA$, then A is said to be **invertible** and B is called the **inverse** of A and it is denoted by A^{-1}.

Thus, $AA^{-1} = I_n = A^{-1}A$

Example 4 : Let $A = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}$, Suppose $B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ such that

$AB = I_n = BA$

$$\Rightarrow \begin{bmatrix} 2\alpha + 7\gamma & 2\beta + 7\delta \\ \alpha + 4\gamma & \beta + 4\delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow 2\alpha + 7\gamma = 1, \quad 2\beta + 7\delta = 0,$

$\alpha + 4\gamma = 0, \quad \beta + 4\delta = 1$

$\Rightarrow \alpha = 4, \beta = -7, \gamma = -1, \delta = 2$ [on solving the equations]

Hence $A^{-1} = B = \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix}$

Theorem 2 : The inverse of a matrix is unique.

Proof : Let A be an invertible matrix and suppose both B and C are inverses of A. Then AB = I = BA, AC = I = CA.

$$\begin{aligned} \therefore B &= BI \\ &= B(AC) \\ &= (BA)C \\ &= IC \\ &= C \end{aligned}$$

Hence the inverse of A is unique.

Theorem 3 : If A, B are invertible matrices, then

i) $(A^{-1})^{-1} = A$

ii) $(AB)^{-1} = B^{-1}A^{-1}$

Proof :

i) Since A is invertible, we have

$$AA^{-1} = I = A^{-1}A$$

$$\Rightarrow A^{-1}A = I = AA^{-1}$$

$$\Rightarrow (A^{-1})^{-1} = A.$$

ii) We have $AA^{-1} = I = A^{-1}A$, $BB^{-1} = I = B^{-1}B$

Now $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= A\,I\,A^{-1}$

$= AA^{-1} = I$

Similarly $(B^{-1}A^{-1})(AB) = I$

Thus $(AB)(B^{-1}A^{-1}) = I = (B^{-1}A^{-1})(AB)$

$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$.

Definition 3 : SINGULAR AND NON-SINGULAR MATRIX

A square matrix A is called a **non-singular** matrix if $|A| \neq 0$, otherwise A is called a **singular** matrix.

Example 5 : Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{bmatrix}$. Then

$$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{vmatrix}$$

$$= 1.(5+2) - 0.(10+1) - 1(4-1) = 4$$

As $|A| \neq 0$, so A is a non-singular matrix.

Take $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix}$. Then $|B| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 0 & 5 \end{vmatrix} = 0$

So, B is a singular matrix.

Theorem 4 : The inverse of a square matrix exists if and only if it is non-singular.

Proof : Let A be a square matrix.

Suppose A^{-1} exists. Then

$AA^{-1} = I = A^{-1}A$

$\Rightarrow |AA^{-1}| = |I| = |A^{-1}A|$

$\Rightarrow |A|.|A^{-1}| = 1$

$\Rightarrow |A| \neq 0$

$\Rightarrow$ A is non-singular.

NOTE : Every matrix is not invertible, only square matrices may have inverse matrices. All square matrices are again not invertible. Only those square matrices whose determinant are non-zero, that is, those square matrices which are non-singular, are invertible.

Conversely, suppose A is non-singular. Then $|A| \neq 0$.

Now, by Theorem 1 we know

$$A(\text{adj}A) = |A|\, I = (\text{adj}A)A$$

$$\Rightarrow \frac{1}{|A|}A(\text{adj}A) = I = \frac{1}{|A|}(\text{adj}A)A, \text{ as } |A| \neq 0$$

$$\Rightarrow A\left(\frac{1}{|A|}\text{adj}A\right) = I = \left(\frac{1}{|A|}\text{adj}A\right)A$$

Hence A^{-1} exists and $A^{-1} = \frac{1}{|A|}\text{adj}A$

This completes the proof of the theorem.

12.5.1 Illustrative Examples on Inverse of Matrices

Example 6 : Show that $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$ are inverses of each other.

Solution : We have $AB = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$$BA = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Thus $AB = I = BA$, and so

$A^{-1} = B,\ B^{-1} = (A^{-1})^{-1} = A.$

Example 7 : Examine the existence of inverse. If exists, find inverse of the matrix.

i) $A = \begin{pmatrix} -2 & 6 \\ 3 & -9 \end{pmatrix}$

ii) $A = \begin{pmatrix} 3 & 5 \\ 2 & 3 \end{pmatrix}$

iii) $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{pmatrix}$

Solution :

i) $|A| = \begin{vmatrix} -2 & 6 \\ 3 & -9 \end{vmatrix} = (-2)(-9) - 6.3 = 18 - 18 = 0$

Since A is a singular matrix, A^{-1} does not exist.

ii) $|A| = \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix} = 9 - 10 = -1$

Since $|A| \neq 0$, A is a singular matrix and so A^{-1} exists,

where $A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{(-1)}\text{adj}A = -\text{adj}A.$

Now $C_{11} = 3$, $C_{12} = -2$, $C_{21} = -5$, $C_{22} = 3$

$\therefore \text{adj}A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}' = \begin{bmatrix} 3 & -2 \\ -5 & 3 \end{bmatrix}' = \begin{bmatrix} 3 & -5 \\ -2 & 3 \end{bmatrix}$

$\therefore A^{-1} = -\begin{bmatrix} 3 & -5 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{vmatrix} = 1.(5+2) - 0.(10+1) - 1(4-1) = 4$

Since $|A| \neq 0$, A^{-1} exists and

$A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{4}\text{adj}A$

$= \frac{1}{4}\begin{bmatrix} 7 & -2 & 1 \\ -11 & 6 & -1 \\ 3 & -2 & 1 \end{bmatrix}$, [see Example 3, 12.4]

$$= \begin{bmatrix} \frac{7}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{11}{4} & \frac{3}{2} & -\frac{1}{4} \\ \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

NOTE : Inverse of a square matrix can be obtained by two methods :

1) Using the formula

$$A^{-1} = \frac{1}{|A|} \text{adj}A$$

for which |A| and adjA should have to enumerate.

2) Using elementary row operations which does not necessitate finding |A| and adjA.

12.5.2 Finding Inverse by Elementary Row Operations

Finding inverse of a square matrix by elementary row operations is based on a theorem stated below (without proof).

Theorem 5 : If a square matrix A of order n is reduced to I_n by successively performing elementary row operations, then the same successive elementary row operations reduces I_n to A^{-1}.

Thus, if we perform successive elementary row operations on $(A \mid I_n)$ which reduces A to I_n, then the resulting matrix is $(I_n \mid A^{-1})$. In symbol, $(A \mid I) \sim (I \mid A^{-1})$.

It should be noted that in performing elementary row operations on (A | I), if we get a zero-row in the A-part, then A does not reduce I and in such case A^{-1} does not exist.

ILLUSTRATIVE EXAMPLES :

Example 8 : Find inverse matrix using elementary row operations :

i) $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{bmatrix}$ ii) $B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$

iii) $C = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & -1 & 1 & 2 \\ -1 & 2 & 1 & -2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$ iv) $D = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{bmatrix}$

Solution :

i) $$(A \mid I) = \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 2 & 6 & -1 & 0 & 1 \end{array}\right], \text{ by } R_2 \to R_2 - 2R_1;\ R_3 \to R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -2 & 1 \end{array}\right], \text{ by } R_3 \to R_3 - 2R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \end{array}\right], \text{ by } R_3 \to \frac{1}{4}R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & -\frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{11}{4} & \frac{3}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \end{array}\right] \text{by } R_1 \to R_1+R_3,\ R_2 \to R_2-R_3$$

$\sim (I \mid A^{-1})$

and hence, $A^{-1} = \begin{bmatrix} \frac{7}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{11}{4} & \frac{3}{2} & -\frac{1}{4} \\ \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$

ii) $(B \mid I) = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right], \text{ by } R_2 \to R_2-2R_1;\ R_3 \to R_3-4R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right], \text{ by } R_3 \leftrightarrow R_3+R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & -1 & 0 & 4 & 0 & -1 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right], \text{ by } R_1 \to R_1+2R_3;\ R_2 \to R_2-R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right], \text{ by } R_2 \to (-1)R_2 \,;\, R_3 \to (-1)R_3$$

$$\sim (I \mid B^{-1})$$

$$\text{and hence, } B^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

iii) $$(C \mid I) = \left[\begin{array}{cccc|cccc} 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 2 & -1 & 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & 2 & 1 & -2 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 & -2 & 1 & 0 & 0 \\ 0 & 3 & 3 & -1 & 1 & 0 & 1 & 0 \\ 0 & -2 & -1 & -2 & -1 & 0 & 0 & 1 \end{array}\right],$$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 + R_1,\ R_4 \to R_4 - R_1$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 3 & 3 & -1 & 1 & 0 & 1 & 0 \\ 0 & -2 & -1 & -2 & -1 & 0 & 0 & 1 \end{array}\right], R_2 \to -\frac{1}{3}R_2$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \end{array}\right],$$

$R_1 \to R_1 - R_2,\ R_3 \to R_3 - 3R_2,\ R_4 \to R_4 + 2R_2$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & -2 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 1 & 0 \end{array}\right], R_3 \leftrightarrow R_4$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 3 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 & \frac{1}{3} & \frac{1}{3} & 0 & -1 \\ 0 & 0 & 1 & -2 & \frac{1}{3} & -\frac{2}{3} & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 \end{array}\right],$$

$R_1 \rightarrow R_1 - R_3$, $R_2 \rightarrow R_2 - R_3$, $R_4 \rightarrow (-1)R_4$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -3 & 4 & 3 & -1 \\ 0 & 1 & 0 & 0 & -\frac{5}{3} & \frac{7}{3} & 2 & -1 \\ 0 & 0 & 1 & 0 & \frac{7}{3} & -\frac{8}{3} & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 \end{array}\right],$$

$R_1 \rightarrow R_1 - 3R_4$, $R_2 \rightarrow R_2 - 2R_4$, $R_3 \rightarrow R_3 + 2R_4$

$\sim (I \mid C^{-1})$

$$\text{and hence, } C^{-1} = \begin{bmatrix} -3 & 4 & 3 & -1 \\ -\frac{5}{3} & \frac{7}{3} & 2 & -1 \\ \frac{7}{3} & -\frac{8}{3} & -2 & 1 \\ 1 & -1 & -1 & 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -9 & 12 & 9 & -3 \\ -5 & 7 & 6 & -3 \\ 7 & -8 & -6 & 3 \\ 3 & -3 & -3 & 0 \end{bmatrix}$$

iv) $$(D \mid I) = \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 1 & 5 & -1 & 0 & 1 & 0 \\ 3 & 13 & -6 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \\ 0 & 4 & 6 & -3 & 0 & 1 \end{array}\right], R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right], \ R_3 \rightarrow R_3 - 2R_2$$

Since the D-part on the r.h.s. has a zero-row, the given matrix is not invertible, i.e., D^{-1} does not exist.

CHECK YOUR PROGRESS

Q.1. Find the adjoint of the following matrices :

i) $\begin{bmatrix} -2 & 3 & 2 \\ 6 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix}$ ii) $\begin{bmatrix} -1 & 1 & 1 \\ 2 & 1 & 1 \\ 5 & -2 & -1 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$

Q.2. For what values of x the following matrices are not invertible?

i) $\begin{pmatrix} 2 & x \\ x & 8 \end{pmatrix}$ ii) $\begin{pmatrix} 2 & -x & 3 \\ 0 & 1 & -1 \\ x & 0 & 1 \end{pmatrix}$

Q.3. If A be a non-singular matrix and B, C are matrices of same order such that the product AB, AC exist and AB = AC, then prove that B = C.

Q.4. Find A^{-1} using adjA :

i) $A = \begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}$ ii) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{pmatrix}$

Q.5. Find A^{-1} using elementary row operations :

i) $A = \begin{bmatrix} -2 & 3 & 2 \\ 6 & 0 & 3 \\ 4 & 1 & -1 \end{bmatrix}$ ii) $A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 & 6 \\ 1 & 1 & 7 \end{bmatrix}$

12.6 RANK AND NULLITY OF A MATRIX

Definition 4 : MINORS OF A MATRIX

Let $A = [a_{ij}]_{m\times n}$ be a given matrix. Let r be a natural number which is less than or equal to the minimum of m, n. By deleting (m–r) rows and (n–r) columns from A, a square matrix of order r can be obtained which is called a square sub-matrix of A. The determinant of this square sub-matrix is called a **Minor** of the matrix A of **order r**. For example,

if $A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & -1 & 1 & 0 \\ 3 & 1 & -2 & 1 \end{bmatrix}$;

then $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$, etc. are minors of order 2.

Similarly, $\begin{bmatrix} 2 & 0 & 4 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$ are minors of order 3.

NOTE : Rank and nullity of a matrix are used in solving a system of linear equations.

Definitions 5 : RANK AND NULLITY OF A MATRIX

Let $A = [a_{ij}]_{m\times n}$ be a given matrix. If all minors of order (r+1) are zero while at least one minor of order r is non-zero, then r is defined as the **rank** of the matrix A, generally denoted by $\rho(A)$.

If $A = [a_{ij}]_{n\times n}$ be a square matrix of rank r, then (n–r) is called the **nullity** of A, generally denoted by $\tau(A)$. Thus

$\rho(A) = r \Rightarrow \tau(A) = n - r$

where n is the order of the square matrix A.

If $A \neq 0$, then A has atleast one non-zero entry and so, atleast one non-zero minor of order 1. Hence, $A \neq 0 \Rightarrow \rho(A) \geq 1$

If A = 0, it has no non-zero entry and so, $\rho(A) = 0$.

Example 9 : Find rank and nullity of :

i) $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ ii) $A = \begin{bmatrix} 4 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 2 & -4 & 3 \\ 2 & -1 & -3 & 5 \\ -1 & 8 & -6 & -1 \end{bmatrix}$

Solution :

i) A has one minor of order 3 which is |A|.

$$\text{Now } |A| = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 1(-1-1) - 2(2-3) - 1(2+3) = -5$$

Thus $|A| \neq 0 \Rightarrow \rho(A) = 3$, $\tau(A) = 3 - 3 = 0$.

ii) Let us first evaluate the minors of order 3 :

$$\begin{vmatrix} 1 & 2 & -4 \\ 2 & -1 & -3 \\ -1 & 8 & -6 \end{vmatrix} = 1.(6+24) - 2(-12-3) - 4(16-1) = 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 5 \\ -1 & 8 & -1 \end{vmatrix} = 1.(1-40) - 2(-2+5) + 3(16-1) = 0$$

$$\begin{vmatrix} 1 & -4 & 3 \\ 2 & -3 & 5 \\ -1 & -6 & -1 \end{vmatrix} = 1.(3+30) + 4(-2+5) + 3(-12-3) = 0$$

$$\begin{vmatrix} 2 & -4 & 3 \\ -1 & -3 & 5 \\ 8 & -6 & -1 \end{vmatrix} = 2(3+20) + 4(1-40) + 3(6+24) = 0$$

Thus all minors of order 3 are zero and so $\rho(A) < 3$.

But $\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1 - 4 = -5$, showing that A has atleast one non-zero minor of order 2. Hence, $\rho(A) = 2$.

Since A is not a square matrix, its nullity is undefined.

iii) The highest order minor is of order 4, which is

$$|A| = \begin{vmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 \\ 2 & 0 & 3 & 0 \end{vmatrix},$$

using $C_2 \to C_2-2C_1$, $C_3 \to C_3+2C_1$, $C_4 \to C_4-3C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 3 & 0 \end{vmatrix} = 1.1.(0-3) = -3 \neq 0$$

$\therefore$ $\rho(A) = 4$ and $\tau(A) = 4 - 4 = 0$.

12.6.1 Finding Rank Using Elementary Row Operations

In example-9 we have seen that to find the rank of a matrix we have to calculate values of a number of determinants, which is a tedious process. We can avoid it by the application of elementary row operations on the matrix reducing it to reduced echelon matrix. This method is based on the following theorem, stated without proof.

Theorem 6 :

a) The rank of a matrix remains invariant under elementary row operations.

b) The rank of a matrix is equal to the rank of its reduced echelon matrix.

c) If a reduced echelon matrix has r non-zero rows, then its rank is r.

From this theorem, it is obvious that if A ~ R where R is the reduced echelon matrix, and if R has r non-zero rows, then $\rho(A) = \rho(R) = r$.

Example 10 : Find the rank of $A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$

Solution : We find the reduced echelon matrix of A performing elementary row operations :

$$A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{8} & \frac{3}{8} & \frac{3}{4} \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}, \text{ by } R_1 \to \frac{1}{8}R_1$$

$$\sim \begin{bmatrix} 1 & \frac{1}{8} & \frac{3}{8} & \frac{3}{4} \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 10 \end{bmatrix}, \text{ by } R_2 \to \frac{1}{3}R_2,\ R_3 \to R_3+8R_1$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{7}{24} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ by } R_1 \to R_1-\frac{1}{8}R_2,\ R_3 \to \frac{1}{10}R_3$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{7}{24} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ by } R_1 \to R_1 - \frac{2}{3}R_3,\ R_2 \to R_2 - \frac{2}{3}R_3$$

The right hand side matrix is in reduced echelon form having 3 non-zero rows. So, $\rho(A) = 3$.

Example 11 : Find the rank of $A = \begin{bmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 6 \end{bmatrix}$

Solution :

$$A = \begin{bmatrix} 2 & 3 & 1 & 0 & 4 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 0 & 2 \\ 3 & 1 & 2 & -1 & 1 \\ 4 & -1 & 3 & -2 & -2 \\ 5 & 4 & 3 & -1 & 6 \end{bmatrix}, R_1 \to \tfrac{1}{2}R_1$$

$$\sim \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 0 & 2 \\ 0 & -\frac{7}{2} & \frac{1}{2} & -1 & -5 \\ 0 & -7 & 1 & -2 & -10 \\ 0 & -\frac{7}{2} & \frac{1}{2} & -1 & -4 \end{bmatrix}, R_2 \to R_2 - 3R_1,$$

$R_3 \to R_3 - 4R_1,\ R_4 \to R_4 - 5R_1$

$$\sim \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} & 0 & 2 \\ 0 & 1 & -\frac{1}{7} & \frac{2}{7} & \frac{10}{7} \\ 0 & -7 & 1 & -2 & -10 \\ 0 & -\frac{7}{2} & \frac{1}{2} & -1 & -4 \end{bmatrix}, R_2 \to -\frac{2}{7}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{5}{7} & -\frac{3}{7} & -\frac{1}{7} \\ 0 & 1 & -\frac{1}{7} & \frac{2}{7} & \frac{10}{7} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, R_1 \to R_1 - \frac{3}{2}R_2,$$

$R_3 \to R_3 + 7R_2$, $R_4 \to R_4 + \frac{7}{2}R_2$

$$\sim \begin{bmatrix} 1 & 0 & \frac{5}{7} & -\frac{3}{7} & 0 \\ 0 & 1 & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, R_1 \to R_1 + \frac{1}{7}R_4, R_2 \to R_2 - \frac{10}{7}R_4$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{5}{7} & -\frac{3}{7} & 0 \\ 0 & 1 & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, R_3 \leftrightarrow R_4$$

The right hand side matrix is in reduced echelon form having 3 non-zero rows.

So, $\rho(A) = 3$.

CHECK YOUR PROGRESS

Q.6. Find rank of the following matrices using definition :

i) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$ ii) $\begin{pmatrix} 3 & 4 & 1 \\ 4 & 3 & 2 \\ 2 & 1 & 4 \end{pmatrix}$ iii) $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Q.7. Find rank of the following matrices using elementary row operations :

i) $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & 4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$

Q.8. Find rank and nullity of $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$

12.7 LET US SUM UP

- If $A = [a_{ij}]_{n\times n}$ be a square matrix, its determinant is denoted by det A or $|A| = |a_{ij}|_{n\times n}$.
- In $A = [a_{ij}]_{n\times n}$, minor of $a_{ij} = |A_{ij}|$, where A_{ij} is the $(n-1)\times(n-1)$ matrix obtained on deleting the i^{th} row and the j^{th} column from A.
- In $A = [a_{ij}]_{n\times n}$, the cofactor of a_{ij} is $C_{ij} = (-1)^{i+j}.|A_{ij}|$.
- If $A = \begin{bmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & d_n \end{bmatrix}$, then $|A| = d_1 d_2 \dots d_n$.

  In particular, $|I_n| = 1$, $n \in N$.
- For two matrices A and B, $|AB| = |A|.|B|$.
- If $A = [a_{ij}]_{n\times n}$, then its adjoint matrix is $\text{adj}A = [C_{ij}]'_{n\times n}$, where C_{ij} is the cofactor of a_{ij}.

- A square matrix A is invertible if there exists another square matrix B such that AB = I = BA. In this case B is called the inverse matrix of A, denoted by A^{-1}.
- The inverse matrix A^{-1} of A is unique.
- If A, B are invertible matrices, then $(A^{-1})^{-1} = A$, $(AB)^{-1} = B^{-1}A^{-1}$.
- A square matrix A is non-singular if $|A| \neq 0$, otherwise it is called singular.
- The inverse of a square matrix exists if and only if it is non-singular.
- For a square matrix A, A(adjA) = |A| I = (adjA)A and hence

  $A^{-1} = \frac{1}{|A|}\text{adj}A$, if A is non-singular.
- The determinant of any square sub-matrix of a given matrix is called a minor of the matrix.
- If a matrix A has atleast one non-zero minor of order r and all other higher order minors are zero, then r is called the rank of A, denoted by $\rho(A)$.
- If A be a square matrix of order n and $\rho(A) = r$, then n–r is called the nullity of A, denoted by $\tau(A)$.

12.8 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 : i) $A = \begin{bmatrix} -2 & 3 & 2 \\ 6 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix}$

$C_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} = -1$, $\quad C_{12} = (-1)^{1+2}\begin{vmatrix} 6 & 3 \\ 4 & -1 \end{vmatrix} = 18$,

$C_{13} = (-1)^{1+3}\begin{vmatrix} 6 & 1 \\ 4 & 0 \end{vmatrix} = -4$, $\quad C_{21} = (-1)^{2+1}\begin{vmatrix} 3 & 2 \\ 0 & -1 \end{vmatrix} = 3$,

$C_{22} = (-1)^{2+2}\begin{vmatrix} -2 & 2 \\ 4 & -1 \end{vmatrix} = -6$, $\quad C_{23} = (-1)^{2+3}\begin{vmatrix} -2 & 3 \\ 4 & 0 \end{vmatrix} = 12$

$C_{31} = (-1)^{3+1}\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = 7,$ $C_{32} = (-1)^{3+2}\begin{vmatrix} -2 & 2 \\ 6 & 3 \end{vmatrix} = 18,$

$C_{33} = (-1)^{3+3}\begin{vmatrix} -2 & 3 \\ 6 & 1 \end{vmatrix} = -20$

$$\text{Hence, adj}A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} -1 & 3 & 7 \\ 18 & -6 & 18 \\ -4 & 12 & -20 \end{bmatrix}$$

ii) & iii) Try yourselves.

Ans. to Q. No. 2 : i) $A = \begin{bmatrix} 2 & x \\ x & 8 \end{bmatrix}$

A is not invertible if $|A| = 0$

i.e., $\begin{bmatrix} 2 & x \\ x & 8 \end{bmatrix} = 0$

i.e., $16 - x^2 = 0$

i.e., $x = \pm 4$.

ii) $A = \begin{bmatrix} 2 & -x & 3 \\ 0 & 1 & -1 \\ x & 0 & 1 \end{bmatrix}$

$|A| = 0 \Rightarrow \begin{bmatrix} 2 & -x & 3 \\ 0 & 1 & -1 \\ x & 0 & 1 \end{bmatrix} = 0$

$\Rightarrow 2(1-0) + x(0+x) + 3(0-x) = 0$

$\Rightarrow x^2 - 3x + 2 = 0$

$\Rightarrow (x-1)(x-2) = 0$

$\Rightarrow x = 1, 2$

So, A is not invertible for $x = 1$ and $x = 2$.

Ans. to Q. No. 3 : Since A is non-singular, A^{-1} exists and $AA^{-1} = I = A^{-1}A$

Now $AB = AC \Rightarrow A^{-1}(AB) = A^{-1}(AC)$

$\Rightarrow (A^{-1}A)B = (A^{-1}A)C$

$\Rightarrow IB = IC$

$\Rightarrow B = C.$

Ans. to Q. No. 4 : i) $A = \begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}$, $|A| = \begin{vmatrix} 1 & 4 \\ 2 & 7 \end{vmatrix} = 7 - 8 = -1$

As $|A| \neq 0$, so A^{-1} exists.

$C_{11} = 7$, $C_{12} = -2$, $C_{21} = -4$, $C_{22} = 1$

$$\text{adj}A = \begin{bmatrix} 7 & -1 \\ -2 & 1 \end{bmatrix}$$

$$\text{Thus, } A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{-1}\begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix}$$

ii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}$

$$|A| = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}$$

$= 1.(21-20) - 2(14-4) + 3(10-3)$

$= 1 - 20 + 21 = 2$

As $|A| \neq 0$, so A^{-1} exists and $A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{2}\text{adj}A$.

Now $C_{11} = 1$, $C_{12} = -10$, $C_{13} = 7$

$C_{21} = 1$, $C_{22} = 4$, $C_{23} = -3$

$C_{31} = -1$, $C_{32} = 2$, $C_{33} = -1$

$$\text{So, adj}A = \begin{bmatrix} 1 & 1 & -1 \\ -10 & 4 & 2 \\ 7 & -3 & -1 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -10 & 4 & 2 \\ 7 & -3 & -1 \end{bmatrix}$$

Ans. to Q. No. 5 : i) $(A \mid I) = \left[\begin{array}{ccc|ccc} -2 & 3 & 2 & 1 & 0 & 0 \\ 6 & 0 & 3 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 & 0 \\ 6 & 0 & 3 & 0 & 1 & 0 \\ 4 & 1 & -1 & 0 & 0 & 1 \end{array}\right], R_1 \to -½R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 & 0 \\ 0 & 9 & 9 & 3 & 1 & 0 \\ 0 & 7 & 3 & 2 & 0 & 1 \end{array}\right], R_2 \to R_2–6R_1, R_3 \to R_3–4R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -\frac{3}{2} & -1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{9} & 0 \\ 0 & 7 & 3 & 2 & 0 & 1 \end{array}\right], R_2 \to \frac{1}{9}R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 0 & \frac{1}{6} & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{9} & 0 \\ 0 & 0 & -4 & -\frac{1}{3} & -\frac{7}{9} & 1 \end{array}\right], R_1 \to R_1+\frac{3}{2}R_2, R_3 \to R_3–7R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 0 & \frac{1}{6} & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{9} & 0 \\ 0 & 0 & 1 & -\frac{1}{12} & -\frac{7}{36} & -\frac{1}{4} \end{array}\right], R_3 \to --\frac{1}{4}R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{24} & \frac{5}{72} & \frac{1}{8} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{12} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{12} & \frac{7}{36} & -\frac{1}{4} \end{array}\right], R_2 \to R_1–½R_3, R_2 \to R_2–R_3$$

$\sim (I \mid A^{-1})$

$$\text{and so, } A^{-1} = \begin{bmatrix} -\frac{1}{24} & \frac{5}{72} & \frac{1}{8} \\ \frac{1}{4} & -\frac{1}{12} & \frac{1}{4} \\ \frac{1}{12} & \frac{7}{36} & -\frac{1}{4} \end{bmatrix} = \frac{1}{72}\begin{bmatrix} -3 & 5 & 9 \\ 18 & -6 & 18 \\ 6 & 14 & -18 \end{bmatrix}$$

ii) $$(A \mid I) = \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & -3 & 6 & 0 & 1 & 0 \\ 1 & 1 & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 3 & 5 & -1 & 0 & 1 \end{array}\right], R_2 \to R_2-2R_1, R_3 \to R_3-R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -1 & 5 & -3 & 1 \end{array}\right], R_1 \to R_1+2R_2, R_3 \to R_3-3R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -5 & 3 & -1 \end{array}\right], R_3 \to (-1)R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 27 & -16 & 6 \\ 0 & 1 & 0 & 8 & -5 & 2 \\ 0 & 0 & 1 & -5 & 3 & -1 \end{array}\right], R_1 \to R_1-6R_3, R_2 \to R_2-2R_3$$

$\sim (I \mid A^{-1})$

$$\text{So, } A^{-1} = \begin{bmatrix} 27 & -16 & 6 \\ 8 & -5 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

Ans. to Q. No. 6 : i) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$

A has no 3rd order minor.

The second order minors are

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0, \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0, \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0$$

Hence, all 2nd order minors are zero.

But $A \neq 0$ and so $\rho(A) = 1$.

ii) $A = \begin{pmatrix} 3 & 4 & 1 \\ 4 & 3 & 2 \\ 2 & 1 & 4 \end{pmatrix}$

3rd order minor is $|A| = \begin{vmatrix} 3 & 4 & 1 \\ 4 & 3 & 2 \\ 2 & 1 & 4 \end{vmatrix}$

$= 3(12–2) – 4(16–4) + 1(4–6)$

$= –20 \neq 0$

So, $\rho(A) = 3$

iii) $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Cleary all 3rd order minors are zero.

It has a non-zero 2nd order minor $\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$ and so, $\rho(A) = 2$.

Ans. to Q. No. 7 : i) $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 2 & 12 & 2 \end{bmatrix}, R_3 \to R_3–R_1, R_4 \to R_4+2R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 \end{bmatrix}, R_3 \to R_3+R_2, R_4 \to R_4–2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, R_3 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, R_3 \rightarrow \frac{1}{16} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, R_1 \rightarrow R_1 - 2R_3,\ R_2 \rightarrow R_2 + 2R_3$$

The right hand side matrix is reduced echelon matrix with three non-zero rows. Hence, $\rho(A) = 3$.

ii) Try yourselves.

Ans. to Q. No. 8 : $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$

$\rho(A) = 3$ [done in 2(i)]

Hence the nullity of A is given by $\tau(A) = 4 - 3 = 1$.

12.9 FURTHER READINGS

1. *Matrix Algebra*, S. K. Jain, Eastern Book House.
2. *Matrices*, A. R. Vasishtha, Krishna Prakashan Mandir, Meerut.
3. *Linear Algebra*, Seymour Lipschutz, Schaum's Solved Problems Series, Tata McGraw Hill.

12.10 MODEL QUESTIONS

Q.1. Find the adjoint of the following matrices

i) $\begin{bmatrix} 3 & -1 \\ 2 & -4 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 5 & -2 & 1 \end{bmatrix}$

Q.2. For the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$, verify that $A(\text{adj}A) = |A|\, I = (\text{adj}A)A$.

Hence, find A^{-1}.

Q.3. Find the adjoint and inverse of $\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Q.4. Find the inverse of $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ verify that $A^3 = A^{-1}$.

Q.5. Given $A = \begin{bmatrix} 3 & -1 & 4 \\ 0 & 2 & 1 \\ 1 & -1 & -2 \end{bmatrix}$, show that $A^3 - 3A^2 - 7A + 18\, I = 0$.

Hence find A^{-1}.

Q.6. Show that A and B are inverses of each other :

(i) $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

(ii) $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$, $B = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}$

Q.7. Find inverse matrix using elementary row operations :

i) $A = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{bmatrix}$ ii) $A = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ iv) $A = \begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

Q.8. If A is α non-singular n×n square matrix, prove that $(A')^{-1} = (A^{-1})'$

Q.9. If A and B are two square matrices of order n×n and AB = I, then show that BA = I.

Q.10. Find rank of the following matrices using definition :

i) $\begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 1 \\ -3 & -1 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

iii) $\begin{bmatrix} 1-\sqrt{6} & \sqrt{3} & \sqrt{2} \\ 2 & \sqrt{6} & -\sqrt{2} \\ 1 & -\sqrt{3} & -\sqrt{6} \end{bmatrix}$

Q.11. Find rank using elementary row operations :

i) $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$

iii) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

And find nullity of the matrices (i) and (iii).

UNIT 13: DETERMINANT-II

UNIT STRUCTURE

13.1 LEARNING OBJECTIVES

After going through this unit, you will be able to:

- define determinant
- evaluate determinants of order 2 and 3
- use the properties of determinants for evaluation of determinants
- solve determinants using Cramer's rule.

13.2 INTRODUCTION

The concept of determinants is a useful tool in solving system of linear equations in two or three variables. In this unit,we shall discusss the concept of determinants.We shall also study many properties of determinants which help in evaluation of determinants. We may use determinants to solve a system of linear equations by a method known as Cramer's rule.

13.3 DETERMINANT OF ORDER 2

Let us consider the equations $a_1x + b_1y = 0$ and $a_2x + b_2y = 0$.

Eliminating x and y from the equations, we have $-\frac{y}{x}=\frac{a_1}{b_1}=\frac{a_2}{b_2}$

$$\Rightarrow a_1b_2-a_2b_1=0$$

which is written in compact form as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=0$

In other words, we have $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=a_1b_2-a_2b_1=0$.

The expression $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called a determinant of the second order. It has two horizotal lines $a_1 \quad b_1$ and $a_2 \quad b_2$ and two vertical lines $\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix}$. The horizontal lines are called rows and the vertical lines are called columns.A determinant of second order has two rows and two columns.The numbers a_1,b_1,a_2,b_2 are called the elements of the determinant. $a_1b_2-a_2b_1$ is called the expansion or the value of the determinant.

Example 5.1: Evaluate (i) $\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix}$ (ii) $\begin{vmatrix} x+1 & x \\ x & x-1 \end{vmatrix}$ (iii) $\begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix}$

(iv) $\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}$

Solution : (i) We have $\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix}=1.(-4)-(-2).3=-4+6=2$.

(ii) $\begin{vmatrix} x+1 & x \\ x & x-1 \end{vmatrix}=(x+1).(x-1)-x.x=x^2-1-x^2=-1$.

(iii) $\begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix}=(x-1)(x^2+x+1)-x^3$

$$=x^3-1-x^3$$
$$=-1\cdot$$

(iv) $\begin{vmatrix} a^2 & ab \\ ab & b^2 \end{vmatrix}=a^2b^2-(ab)^2=0$.

Example 5.2 : Find the value of x if

(i) $\begin{vmatrix} x-3 & x \\ x+1 & x+3 \end{vmatrix} = 6$ (ii) $\begin{vmatrix} 2x-1 & 2x+1 \\ x+1 & 4x+2 \end{vmatrix} = 0$

Solution : (i) We have $\begin{vmatrix} x-3 & x \\ x+1 & x+3 \end{vmatrix} = (x-3)(x+3) - x(x+1)$

$$= (x^2 - 9) - x^2 - x$$
$$= -x - 9$$

$$\therefore -x - 9 = 6$$
$$\Rightarrow -x = 15$$
$$\Rightarrow x = -15$$

(ii) We have $\begin{vmatrix} 2x-1 & 2x+1 \\ x+1 & 4x+2 \end{vmatrix} = (2x-1)(4x+2) - (x+1)(2x+1)$

$$= 8x^2 + 4x - 4x - 2 - 2x^2 - x - 2x - 1$$
$$= 6x^2 - 3x - 3$$
$$= 3(2x^2 - x - 1)$$

$$\therefore 3(2x^2 - x - 1) = 0$$
$$\Rightarrow 2x^2 - x - 1 = 0$$
$$\Rightarrow x = \frac{1 \pm \sqrt{1+8}}{4}$$
$$= \frac{1 \pm 3}{4}$$
$$= 1, -\frac{1}{2}$$

Q 5.3. Solve for x if $\begin{vmatrix} x & 5 \\ 7 & x \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} = 0$.

Solution : Given, $\begin{vmatrix} x & 5 \\ 7 & x \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} = 0$

$$\Rightarrow (x^2 - 35) + (1 - 2) = 0$$
$$\Rightarrow x^2 - 35 - 1 = 0$$
$$\Rightarrow x^2 - 36 = 0$$
$$\Rightarrow x^2 = 36$$

$$\Rightarrow x = \pm 6$$

CHECK YOUR PROGRESS

Q 1: Evaluate :

(i) $\begin{vmatrix} 2 & -3 \\ 7 & 11 \end{vmatrix}$ (ii) $\begin{vmatrix} 2+\sqrt{3} & 3+\sqrt{11} \\ 3-\sqrt{11} & 2-\sqrt{3} \end{vmatrix}$

Q 2: Solve for x :

(i) $\begin{vmatrix} 2 & 3 \\ 1 & 4x \end{vmatrix} = \begin{vmatrix} 2x & -1 \\ 5 & x \end{vmatrix}$ (ii) $\begin{vmatrix} x & 3 \\ 4 & x \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ 2x & 5 \end{vmatrix}$

Q 3: If $\begin{vmatrix} 3 & p \\ 4 & 6 \end{vmatrix} = 6$, find the value of p .

Q 4: Prove that $\begin{vmatrix} a+ib & c+id \\ c-id & a-ib \end{vmatrix} = a^2 + b^2 - c^2 - d^2$

13.4 DETERMINANT OF ORDER 3

In 5.3, We have already discussed determinant of order 2. Now, we think of a determinant which has 3 rows and 3 colums.

Let us consider the equations

$a_1x + b_1y + c_1z = 0$ ... (1)

$a_2x + b_2y + c_2z = 0$ ... (2)

and $a_3x + b_3y + c_3z = 0$ ... (3)

Soving (2) and (3), we get $\dfrac{x}{b_2c_3 - b_3c_2} = \dfrac{y}{c_2a_3 - c_3a_2} = \dfrac{z}{a_2b_3 - a_3b_2}$

Substituting these proportional values of x, y and z in (1), we get

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$$

which can be written in compact form as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

In other words, we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) + b_1(a_3c_2 - a_2c_3) + c_1(a_2b_3 - a_3b_2)$$

The expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called a determinant of the third order. Since it has 3 rows and 3 columns,it is called a determinant of order 3.

$a_1(b_2c_3 - b_3c_2) + b_1(a_3c_2 - a_2c_3) + c_1(a_2b_3 - a_3b_2)$ is called the expansion or the value of the determinant.

Note : A determinant of order 3 has 9 elements.

Value of a determinant :

Determinant of order three can be determined by expressing it in terms of second order determinants.This is known as expansion of a determinant along a row (or a column).There are six ways of expanding a determinant of order 3 corresponding to each of three rows (R_1, R_2 and R_3) and three columns (C_1, C_2 and C_3) giving the same value.

Consider the determinant of order 3

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

First we expand the given determinant along first row (R_1):

Step 1 : We multiply element a_1 of R_1 by $(-1)^{1+1}$ [$(-1)^{\text{position of the element } a_1}$] and with the second order determinant obtained by deleting the elements of first row (R_1) and first column (C_1) as a_1 lies in R_1 and C_1.

i.e., $(-1)^{1+1} a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

Step 2 : We multiply second element b_1 of R_1 by $(-1)^{1+2}$ [$(-1)^{\text{position of the element } b_1}$] and with the second order determinant obtained by deleting the elements of first row (R_1) and second column (C_2) as b_1 lies in R_1 and C_2.

i.e., $(-1)^{1+2} b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$

Step 3 : We multiply third element c_1 of R_1 by $(-1)^{1+3}$ [$(-1)^{\text{position of the element } c_1}$] and with the second order determinant obtained by deleting the elements of first row(R_1) and third column (C_3) as c_1 lies in R_1 and C_3.

i.e., $(-1)^{1+3} c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

Step 4 : The expansion of determinant is written as the sum of all the three terms obtained in step 1, 2 and 3 above and is given by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1)^{1+1} a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + (-1)^{1+2} b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + (-1)^{1+3} c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 .$$

Expansion along second row (R_2):

Consider the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Expanding along R_2, we get

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1)^{2+1} a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + (-1)^{2+2} b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + (-1)^{2+3} c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$= -a_2(b_1c_3 - b_3c_1) + b_2(a_2c_3 - a_3c_1) - c_2(a_1b_3 - a_3b_1)$$

$$= -a_2b_1c_3 + a_2b_3c_1 + a_1b_2c_3 - a_3b_2c_1 - a_1b_3c_2 + a_3b_1c_2 .$$

Expanding along C_1, we get

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1)^{1+1} a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + (-1)^{2+1} a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + (-1)^{3+1} a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

Similarly, we can expand the determinant along R_3, C_2 and C_3.

Note : Expanding a determinant along any row or column gives same value.

Example 5.4 : Evaluate the determinant $\begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$

Solution : Expanding along R_1 we get

$$\begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix} = 1\begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} - 2\begin{vmatrix} -1 & 0 \\ 4 & 0 \end{vmatrix} + 4\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix}$$

$$= 1(0-0) - 2(0-0) + 4(-1-12)$$

$$= -52\cdot$$

Example 5.5 : Evaluate the determinant $\Delta = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & 4 \\ 1 & -1 & -2 \end{vmatrix}$

(i) by expanding about any row

(ii) by expanding about any column.

Solution : (i) (a) Expanding about the first row

$$\Delta = 1\begin{vmatrix} 3 & 4 \\ -1 & -2 \end{vmatrix} - (-2)\begin{vmatrix} -2 & 4 \\ 1 & -2 \end{vmatrix} + 3\begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix}$$

$$= 1[3(-2) - (-1)4] + 2[(-2)(-2) - (1)(4)] + 3[(-2)(-1) - (3)(1)]$$

$$= -6 + 4 + 2(4-4) + 3(2-3)$$

$$= -2 - 3$$

$$= -5$$

(b) Expanding about the second row

$$\Delta = (-1)(-2)\begin{vmatrix} -2 & 3 \\ -1 & -2 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} - 4\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix}$$

$$= 2(4+3) + 3(-2-3) - 4(-1+2)$$

$$= 14 - 15 - 4$$

$$= -5$$

(c) Expanding about the third row

$$\Delta = 1\begin{vmatrix} -2 & 3 \\ 3 & 4 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix} + 2\begin{vmatrix} 1 & -2 \\ -2 & 3 \end{vmatrix}$$

$= -8 - 9 + 4 + 6 - 2(3 - 4)$

$= -17 + 10 + 2$

$= -5$

(ii) (a) Expanding about first column

$$\Delta = 1\begin{vmatrix} 3 & 4 \\ -1 & -2 \end{vmatrix} + 2\begin{vmatrix} -2 & 3 \\ -1 & -2 \end{vmatrix} + 1\begin{vmatrix} -2 & 3 \\ 3 & 4 \end{vmatrix}$$

$= -6 + 4 + 2(4 + 3) + 1(-8 - 9)$

$= -2 + 14 - 17$

$= -5$

(b) Expanding about the second column

$$\Delta = (-1)(-2)\begin{vmatrix} -2 & 4 \\ 1 & -2 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} + (-1)(-1)\begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix}$$

$= 2(4 - 4) + 3(-2 - 3) + (4 + 6)$

$= -15 + 10$

$= -5$

(c) Expanding about the third column

$$\Delta = 3\begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} - 4\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} + (-2)\begin{vmatrix} 1 & -2 \\ -2 & 3 \end{vmatrix}$$

$= 3(2 - 3) - 4(-1 + 2) - 2(3 - 4)$

$= -3 - 4 + 2$

$= -5$

13.5 PROPERTIES OF DETERMINANTS

There are some properties of determinants, which are very much useful in solving problems. Here we are going to discuss the properties only for the determinant of order 3.

> The system of writing equations within [ ] is known as Matrix. We shall discuss Matrix algebra in the next unit.

Property 1 : The value of determinant remains unchanged if a rows and columns are interchanged.

i.e., $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Note : If R_i = *i*th row and C_i = *i*th column, then for interchange of row and columns, we will symbolically write $C_i \leftrightarrow R_i$.

Property 2 : If any two adjacent rows (or columns) of a determinant are interchanged, then sign of determinant changes.

i.e., $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Note : We can denote the interchange of rows by $R_i \leftrightarrow R_j$ and interchange of columns by $C_i \leftrightarrow C_j$.

Property 3 : If any two rows (or columns) of a determinant are identical (all corresponding elements are same), then value of determinant is zero.

i.e., $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Property 4 : If each element of a row (or a column) of a determinant is multiplied by a constant *k*, then its value gets multiplied by *k*.

i.e., $$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Note :The value of the determinant remains unchanged.by applying $R_i \leftrightarrow kR_i$ or $C_i \leftrightarrow kC_i$ to the determinant .

Property 5 : If to any row(or column) is added k times the corresponding elements of another row (or column), the value of the determinant remains unchanged.

i.e., $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+ka_2 & b_1+kb_2 & c_1+kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Note : The value of determinant remain same if we apply the operation $R_i \leftrightarrow R_i + kR_j$ or $C_i \leftrightarrow C_i + k\,C_j$.

Property 6 : If any row (or column) is the sum of two or more elements, then the determinant can be expressed as sum of two or more determinants.

i.e., $\begin{vmatrix} a_1+d_1 & b_1+d_2 & c_1+d_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & d_2 & d_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Illustrative Examples :

Example 5.6 : Prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Solution : $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$

(Apply $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$)

$$= (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix}$$

$$= (b-a)(c-a)(c+a-b-a)$$
$$= (b-a)(c-a)(c-b)$$
$$= (a-b)(b-c)(c-a).$$

Example 5.7 : Show that

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$$

Solution : $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

Apply $C_3 \to C_3 + C_2$

$$= \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & b+c+a \\ 1 & c & c+a+b \end{vmatrix}$$

$$=(a+b+c)\begin{vmatrix}1 & a & 1\\ 1 & b & 1\\ 1 & c & 1\end{vmatrix}$$

$=(a+b+c).0$ (C_1 and C_3 are identical)

$=0\cdot$

Example 5.8 : Find x if $\begin{vmatrix}1 & x & -4\\ 5 & 3 & 0\\ -2 & -4 & 8\end{vmatrix}=0$.

Solution : Expanding by 1st row

$$\begin{vmatrix}1 & x & -4\\ 5 & 3 & 0\\ -2 & -4 & 8\end{vmatrix}=1\begin{vmatrix}3 & 0\\ -4 & 8\end{vmatrix}-x\begin{vmatrix}5 & 0\\ -2 & 8\end{vmatrix}+(-4)\begin{vmatrix}5 & 3\\ -2 & -4\end{vmatrix}$$

$$=1(24)-x(40)-4(-20+6)$$

$$=24-40x+56$$

$$=-40x+80$$

$\Rightarrow -40x+80=0$

$\Rightarrow x=2\cdot$

Example 5.9 : Solve for x if $\begin{vmatrix}0 & 1 & 0\\ x & 2 & x\\ 1 & 3 & x\end{vmatrix}=0$.

Solution : Given , $\begin{vmatrix}0 & 1 & 0\\ x & 2 & x\\ 1 & 3 & x\end{vmatrix}=0$

$\Rightarrow 0\begin{vmatrix}2 & x\\ 3 & x\end{vmatrix}-1\begin{vmatrix}x & x\\ 1 & x\end{vmatrix}+0\begin{vmatrix}x & 2\\ 1 & 3\end{vmatrix}=0$

$\Rightarrow 0-1(x^2-x)+0=0$

$\Rightarrow -x^2+x=0$

$\Rightarrow x(1-x)=0$

$\Rightarrow x=0,1$.

CHECK YOUR PROGRESS

Q 5: Find the value of the following determinants:

(a) $\begin{vmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{vmatrix}$ (b) $\begin{vmatrix} 4 & -1 & 5 \\ 5 & 7 & -3 \\ 3 & 4 & 1 \end{vmatrix}$

(c) $\begin{vmatrix} 1 & -1 & 2 \\ 4 & 1 & 1 \\ 5 & -1 & 8 \end{vmatrix}$

(i) by expanding about any one row

(ii) by expanding about any one column.

Q 6: Factorise the determinants and solve the equations :

(a) $\begin{vmatrix} x & x^2 & x^2 \\ 3 & 9 & 27 \\ -1 & -2 & -3 \end{vmatrix} = 0$ (b) $\begin{vmatrix} 1 & x^3 & 2 \\ 1 & x^2 & 3 \\ 1 & x & 4 \end{vmatrix} = 0$

13.6 SOLUTION OF A SET OF LINEAR EQUATIONS BY CRAMER'S RULE

Let the system of n non-homogenous linear equations in n-unknowns linear

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots + a_{1n}x_n = b_1 \rightarrow$ (1)

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots + a_{2n}x_n = b_2 \rightarrow$ (2)

....

$a_{n1}x_1 + a_{n2}x_2 + \ldots . + a_{nn}x_n = b_n \rightarrow$ (n)

The system can be written as –

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix}_{n\times 1}$$

i.e. AX = B

Determinant of the co-efficient matrix A

$= |A| = D$ (say)

Multiplying the equations (1), (2), ..., (n) respectively by the co-factors of $a_{11}, a_{21}, \cdots$ i.e. $A_{11}, A_{21}, \cdots, A_{n1}$ and adding we get

$$A_{11}(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n) + A_{21}(a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n) + \cdots$$
$$+ A_{n1}(a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n) = b_1A_{11} + b_2A_{21} + \cdots + b_nA_{n1}$$
$$\Rightarrow (a_{11}A_1 + a_{21}A_{21} + \cdots + a_{n1}A_{n1})x_1 = b_1A_{11} + b_2A_{21} + \cdots + b_nA_{n1}$$
$$\Rightarrow Dx_1 = D_1$$

Where

$$D = \begin{vmatrix} a_{11} & a_{12} \text{........} & a_{1n} \\ a_{21} & a_{22} \text{........} & a_{2n} \\ \text{..........................} & & \\ a_{n1} & a_{n2} \text{........} & a_{nn} \end{vmatrix} \text{ and } D_1 = \begin{vmatrix} b_1 & a_{12} \text{........} & a_{1n} \\ b_2 & a_{22} \text{........} & a_{2n} \\ \text{..........................} & & \\ b_n & a_{n2} \text{........} & a_{nn} \end{vmatrix}$$

D_1 is the determinant obtained from D by replacing the elements of 1st column by corresponding b's

$$\Rightarrow \frac{x_1}{D_1} = \frac{1}{D} \text{ Provided } D \neq 0$$

Similarly multiplying the equations (1), (2), ... (n) by co-factors of the elements of 2nd column of |A| and adding, we get –

$$Dx_2 = D_2 \quad \text{where} \quad D_2 = \begin{vmatrix} a_{11} & b_1 \text{.....} a_{1n} \\ a_{21} & b_2 \text{.....} a_{2n} \\ a_{n1} & b_n \text{.....} a_{nn} \end{vmatrix}$$

$$\Rightarrow \frac{x_2}{D_2} = \frac{1}{D}, D \neq 0$$

As above we will get

$$\frac{x_1}{D_1} = \frac{x_2}{D_2} = \text{.........} = \frac{x_n}{D_n} = \frac{1}{D}, D \neq 0$$

The unique solution of the given system of equation provided $D \neq 0$ in the coefficient matrix is non-singular.

i.e. the rank of the co-efficient matrix is n = number of variables.

Note : For a system of n non-homogeneous linear equations with n-unknowns

$$\frac{x_1}{D_1} = \frac{x_2}{D_2} \text{.........} \frac{x_n}{D_n} = \frac{1}{D}, D \neq 0$$

Example 5.10 : Solve the equations

3x + y +2z = 3

2x - 3y – z = –3

x + 2y + z = 4

Using Cramer's rule.

Solution : We have

$$D = |A| = \begin{vmatrix} 3 & 1 & -2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

= 3(–3 + 2) – 1(2+1) + 2(4+3)
= –3 –3 + 14
= $8 \neq 0$

$$D_1 = \begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix}$$

= 3(–3+2) –1 (–3+4) + 2(–6+12)
= –3 – 1 + 12
= 8

$$D_2 = \begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix}$$

$$= 3(-3+4) - 3(2+1) + 2(8+3)$$
$$= 3 - 9 + 22 = 16$$

$$D_3 = \begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix}$$

= 3 (–12+6) –1(8+3) + 3 (4+3)
= -18 – 11 + 21
= –8

$$\therefore x = \frac{D_1}{D} = \frac{8}{8} = 1$$

$$y = \frac{D_2}{D} = \frac{16}{8} = 2$$

$$z = \frac{D_3}{D} = \frac{-8}{8} = -1$$

Example 5.11 : Solve the equations using Cramer's rule:

x+2y+3z = 6

3x–2y+z=2

4x+2y+z=7

Solution :

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{vmatrix} \begin{aligned} &= 1(-2-2)-2(3-4)+3(6+8) \\ &= -4+2+42 = 40 \neq 0 \end{aligned}$$

$$D_1 = \begin{vmatrix} 6 & 2 & 3 \\ 2 & -2 & 1 \\ 7 & 2 & 1 \end{vmatrix} \begin{aligned} &= 6(-2-2)-2(2-7)+3(4+14) \\ &= -24+10+54 = 40 \end{aligned}$$

$$D_2 = \begin{vmatrix} 1 & 6 & 3 \\ 3 & 2 & 1 \\ 4 & 7 & 1 \end{vmatrix}$$

= 1(2–7)–6(3–4)+3(21–8)
= –5+6+39
= 40

$$D_3 = \begin{vmatrix} 1 & 2 & 6 \\ 3 & -2 & 2 \\ 4 & 2 & 7 \end{vmatrix}$$

= 1(–14–4)–2(21–8)+6(6+8)
= –18–26+84
= 84 – 44
= 40

$$\therefore x = \frac{D_1}{D} = \frac{40}{40} = 1$$

$$y = \frac{D_2}{D} = \frac{40}{40} = 1$$

$$z = \frac{D_3}{D} = \frac{40}{40} = 1$$

CHECK YOUR PROGRESS

Solve the following system of equations by using Cramer's rule :

Q 7: $3x + 5y = 8$, $-x+2y-z=0$, $3x-6y+4z=1$

Q 8: $x_1+x_2+x_3=7$, $x_1 - x_2 + x_3 = 2$, $2x_1 - x_2 + 3x_3 = 9$

13.7 LET US SUM UP

- The value of determinant is unaltered, when its rows and columns are interchanged.
- If any two adjacent rows (columns) of a determinant are interchanged, then the value of the determinant changes only in sign.
- If the determinant has two identical rows (columns), then the value of the determinant is zero.
- If all the elements in a row or in a (column) of a determinant are multiplied by a constant $k(k > 0)$ then the value of the determinant is multiplied by k.
- The value of the determinant is unaltered when a constant multiple of the elements of any row (column), is added to the corresponding elements of a different row (column) in a determinant.
- If each element of a row (column) of a determinant is expressed as the sum of two or more terms, then the determinant is expressed as the sum of two or more determinants of the same order.

13.8 FURTHER READING

1) Agarwal, D.K. (2012). *Business Mathematics*, New Delhi: Vrindra Publication (p) Ltd.

2) Baruah, S. (2011). *Basic Mathematics and Its Application in Economics*, New Delhi: Trinity Press Pvt. Ltd.

3) Bose, D. (2004). *Mathematical Economics;* New Delhi: Himalaya

Publishing House.

4) Chiang, A.C. (2006) *Fundamental Methods of Economics Analysis*; New Delhi: MC Graw Hill Education India.

5) Kandoi, Balwant (2011). *Mathematics for Business and Economics with Application;* New Delhi: Himalaya Publishing House.

13.9 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: (i) $\begin{vmatrix} 2 & -3 \\ 7 & 11 \end{vmatrix} = 22 + 21$

$$= 43.$$

(ii) $\begin{vmatrix} 2+\sqrt{3} & 3+\sqrt{11} \\ 3-\sqrt{11} & 2-\sqrt{3} \end{vmatrix} = \left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right) - \left(3+\sqrt{11}\right)\left(3-\sqrt{11}\right)$

$$= (4-3) - (9-11)$$

$$= 3.$$

Ans to Q No 2: (i) Given, $\begin{vmatrix} 2 & 3 \\ 1 & 4x \end{vmatrix} = \begin{vmatrix} 2x & -1 \\ 5 & x \end{vmatrix}$

$$\Rightarrow 8x - 3 = 2x^2 + 5$$

$$\Rightarrow 2x^2 - 8x + 8 = 0$$

$$\Rightarrow x^2 - 4x + 4 = 0$$

$$\Rightarrow (x-2)^2 = 0$$

$$\Rightarrow x = 2,2.$$

(ii) Given, $\begin{vmatrix} x & 3 \\ 4 & x \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ 2x & 5 \end{vmatrix}$

$$\Rightarrow x^2 - 12 = 0 + 4x$$

$$\Rightarrow x^2 - 4x - 12 = 0$$

$$\Rightarrow x^2 - 6x + 2x - 12 = 0$$

$$\Rightarrow x(x-6) + 2(x-6) = 0$$

$$\Rightarrow (x-6)(x+2) = 0$$

$$\Rightarrow x = -2,6.$$

Ans to Q No 3: Given, $\begin{vmatrix} 3 & p \\ 4 & 6 \end{vmatrix} = 6$

$\Rightarrow 18 - 4p = 6$

$\Rightarrow -4p = -12$

$\Rightarrow p = 3$.

Ans to Q No 4: L.H.S $= \begin{vmatrix} a+ib & c+id \\ c-id & a-ib \end{vmatrix}$

$= (a+ib)(a-ib) - (c+id)(c-id)$

$= \{a^2 - (ib)^2\} - \{c^2 - (id)^2\}$

$= a^2 + b^2 - c^2 - d^2$

Ans to Q No 5: (a) (i) Expanding about the first row

$$\begin{vmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 2\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$$

$= 2(2-1) - 1(-1-2) - 3(1+4)$

$= 2 + 3 - 15$

$= -10$

(ii) Expanding about the first column

$$\begin{vmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 2\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & -3 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix}$$

$= 2(2-1) - 1(-1+3) + 2(1-6)$

$= 2 - 2 - 10$

$= -10$.

(b) Try yourself

(c) Try yourself.

Ans to Q No 6: (a) Given, $\begin{vmatrix} x & x^2 & x^2 \\ 3 & 9 & 27 \\ -1 & -2 & -3 \end{vmatrix} = 0$

$$\Rightarrow 3x\begin{vmatrix} 1 & x & x^2 \\ 1 & 3 & 9 \\ -1 & -2 & -3 \end{vmatrix} = 0$$

$$\Rightarrow 3x\begin{vmatrix} 3 & 9 \\ -2 & -3 \end{vmatrix} - 3x\begin{vmatrix} x & x^2 \\ -2 & -3 \end{vmatrix} - 3x\begin{vmatrix} x & x^2 \\ 3 & 9 \end{vmatrix} = 0$$

$$\Rightarrow 3x(-9+18) - 3x(-3x+2x^2) - 3x(9x-3x^2) = 0$$

$$\Rightarrow 3x(9+3x-2x^2-9x+3x^2) = 0$$

$$\Rightarrow 3x(x^2-6x+9) = 0$$

$$\Rightarrow 3x(x-3)^2 = 0$$

$$\Rightarrow x = 0,3$$

(b) Try yourself.

Answer Q No 7:

3x + 5y + 0z = 8

–x + 2y – z = 0

3x – 6y + 4z = 1

$$D = \begin{vmatrix} 3 & 5 & 0 \\ -1 & 2 & -1 \\ 3 & -6 & 4 \end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & -1 \\ -6 & 4 \end{vmatrix} - 5\begin{vmatrix} -1 & -1 \\ 3 & 4 \end{vmatrix}$$

= 3 (8-6) – 5(–4+3)

= 6+5 = $11 \neq 0$

$$D_1 = \begin{vmatrix} 8 & 5 & 0 \\ 0 & 2 & -1 \\ 1 & -6 & 4 \end{vmatrix} = 8\begin{vmatrix} 2 & -1 \\ -6 & 4 \end{vmatrix} - 5\begin{vmatrix} 0 & -1 \\ 1 & 4 \end{vmatrix}$$

= 8(8–6) –5x1=16–5=11

$$D_2 = \begin{vmatrix} 3 & 8 & 0 \\ -1 & 0 & -1 \\ 3 & 1 & 4 \end{vmatrix}$$

$$= 3\begin{vmatrix} 0 & -1 \\ 1 & 4 \end{vmatrix} - 8\begin{vmatrix} -1 & -1 \\ 3 & 4 \end{vmatrix}$$

= 3(0+1)–8(–4+3)

= 3+8 = 11

$$D_3 = \begin{vmatrix} 3 & 5 & 8 \\ -1 & 2 & 0 \\ 3 & -6 & 1 \end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & 0 \\ -6 & 1 \end{vmatrix} - 5\begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix} + 8\begin{vmatrix} -1 & 2 \\ 3 & -6 \end{vmatrix}$$

= 3(2–0) –5(–1–0) +8(6–6)

= 6+5 =11

$$\therefore x = \frac{D_1}{D} = \frac{11}{11} = 1$$

$$y = \frac{D_2}{D} = \frac{11}{11} = 1$$

$$y = \frac{D_3}{D} = \frac{11}{11} = 1$$

i.e. x =1, y =1, z =1.

Answer Q No 8:

$x_1 + x_2 + x_3 = 6$
$x_1 - x_2 + x_3 = 2$
$2x_1 - x_2 + 3x_3 = 9$

$$\therefore D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 3 \end{vmatrix} = 1x\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} - 1x\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + a\,x\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix}$$

= (–3+1)–(3–2)+(–1+2)

= –2–1+1= $-2 \neq 0$

$$\therefore D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ 9 & -1 & 3 \end{vmatrix} = 6\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} - 1x\begin{vmatrix} 2 & 1 \\ 9 & 3 \end{vmatrix} + 1x\begin{vmatrix} 2 & -1 \\ 9 & -1 \end{vmatrix}$$

= 6(–3+1)–(6–9)+(–2+9)
= –12+3+7= – 2

$$D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 2 & 9 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 \\ 9 & 3 \end{vmatrix} - 6x\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 9 \end{vmatrix}$$

= (6–9) –6(3–2) +(9–4)
= –3 –6 +5
= –9 + 5

$= -4$

$$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 2 & -1 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 2 \\ -1 & 9 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 2 & 9 \end{vmatrix} + 6\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix}$$

$= (-9+2) -(9-4) + 6 \text{ x}(-1+2)$

$= -7-5+6$

$= -12+6$

$= -6$

$$x_1 = \frac{D_1}{D} = \frac{-2}{-2} = 1$$

$$x_2 = \frac{D_2}{D} = \frac{-4}{-2} = 2$$

$$x_3 = \frac{D_3}{D} = \frac{-6}{-2} = 3$$

13.10 MODEL QUESTIONS

Q 1: Evaluate (i) $\begin{vmatrix} 4 & 6 \\ -2 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix}$ (iii) $\begin{vmatrix} -2 & -4 \\ -1 & -6 \end{vmatrix}$

Q 2: Evaluate (i) $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \\ 1 & 2 & 4 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Q 3: Show that

$$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

Q 4: Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-a)(c-a)(a-b)(a+b+c)$$

Q 5: Show that $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$.

Q 6: Show that $\begin{vmatrix} a & b & c \\ a-c & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3+b^3+c^3-3abc$

Q 7: Prove that $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$

Q 8: Show that $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy$

Q 9: Without expanding the determinant, prove that

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

Q 10: Solve using Cramer's rule

$$2x+3y+z=9$$
$$x+2y+3z=6$$
$$3x+y+2x=8$$

*** ***** ***

UNIT 14 : SYSTEM OF LINEAR EQUATIONS

UNIT STRUCTURE

14.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- write a system of linear equations in matrix form
- know condition for existence of unique solution of *n* linear equations in *n* unknowns.
- solve *n* linear equations in *n unknowns by Cramer's rule*
- know consistent and inconsistent system of *m* linear equations in *n* unknowns
- solution of linear equations by Gaussian elimination method.

14.2 INTRODUCTION

We are familiar with the method of solution of two linear equations in two variables. The equations ax + by = c, px + qy = d can easily be solved by the method of elimination of one of the two variables. But the method is difficult to work with when the number of variables and number of equations are more than two. It becomes more difficult when the number of equations and the number of variables are not equal. In this unit we shall discuss method of solution of such a system of linear equations, converting the system of equations to matrix form and then solving by applying Cramer's rule and the Gaussian elimination method.

14.3 A SYSTEM OF n LINEAR NON-HOMOGENEOUS EQUATIONS IN *n* UNKNOWNS

A system of equations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \quad \dots\dots\dots (1)$$
$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

is called a system of *n* **non-homogeneous linear equations** in *n* unknowns [where b_i, $a_{ij} \in$ R or any field F].

Let A be the matrix of the coefficients of the unknowns, X be the column matrix of the unknowns and B be the column matrix of the right hand side constants. That is,

$$A = [a_{ij}]_{n\times n},\ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\ B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

It can be easily verified that the system (1) can be written as a matrix equation AX = B ....(2)

For example, the equations $x - z = 1$

$2x + y - z = 1$

$x + 2y + 5z = 2$

can be written as AX = B, where

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

14.3.1 Condition for Existence of Unique Solution

Theorem : The system of equations AX = B has a unique solution if $|A| \neq 0$, i.e., A is a non-singular matrix.

Proof : Let $|A| \neq 0$. Then A^{-1} exists and $AA^{-1} = I = A^{-1}A$ ......(1)

Now $AX = B$

$\Leftrightarrow A^{-1}(AX) = A^{-1}B$

$\Leftrightarrow (A^{-1}A)X = A^{-1}B$

$\Leftrightarrow IX = A^{-1}B$, by (1)

$\Leftrightarrow X = A^{-1}B$.

This shows that $A^{-1}B$ is a solution of the system AX = B.

To prove uniqueness, let us assume that X_1 and X_2 are two solutions of the system.

Then $AX_1 = B$, $AX_2 = B$

$\Leftrightarrow AX_1 = AX_2$

$\Leftrightarrow A^{-1}(AX_1) = A^{-1}(AX_2)$

$\Leftrightarrow (A^{-1}A)X_1 = (A^{-1}A)X_2$

$\Leftrightarrow IX_1 = IX_2$, by (1)

$\Leftrightarrow X_1 = X_2$.

This shows that the system AX = B has a unique solution.

Example : Let us consider the equations

$x - z = 1,$

$2x + y - z = 1,$

$x + 2y + 5z = 2.$

The matrix form is AX = B where

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

$$|A| = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{bmatrix} = 1.(5+2) - 0.(10+1) - 1(4-1) = 4$$

Since $|A| \neq 0$, the system has a unique solution given by

$$X = A^{-1}B = \frac{1}{|A|}.(\text{adj } A)B$$

$$= \frac{1}{4}(\text{adj } A)B$$

$$\text{Now, adjA} = \begin{bmatrix} 7 & -2 & 1 \\ -11 & 6 & -1 \\ 3 & -2 & 1 \end{bmatrix} \text{ (workout yourselves)}$$

$$\therefore X = \frac{1}{4}\begin{bmatrix} 7 & -2 & 1 \\ -11 & 6 & -1 \\ 3 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 7 & -2 & +2 \\ -11 & +6 & -2 \\ 3 & -2 & +2 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} \\ -\frac{7}{4} \\ \frac{3}{4} \end{bmatrix}$$

$$\text{Thus } = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{7}{4} \\ -\frac{7}{4} \\ \frac{3}{4} \end{bmatrix}$$

Hence $x = \frac{7}{4}$, $y = -\frac{7}{4}$, $z = \frac{3}{4}$.

14.3.2 Solution by Cramer's Rule

Let us counsider the *n* equations in *n* unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots & \quad \ldots\ldots\ldots (1) \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots & \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

In matrix notation, the system of equations is equivalent to

$$AX = B$$

Where $A = [a_{ij}]_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If $|A| \neq 0$, then by theorem 1 the unique solution is given by

$$X = A^{-1}B = \frac{1}{|A|}(\text{adj } A)B$$

$$= \frac{1}{|A|}\begin{bmatrix} c_{11} & c_{21} & \ldots & c_{n1} \\ c_{12} & c_{22} & \ldots & c_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ c_{1n} & c_{2n} & \ldots & c_{nn} \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

where c_{ij} = cofactor of a_{ij} in A.

Thus, $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} c_{11}b_1 + c_{21}b_2 + \ldots + c_{n1}b_n \\ c_{12}b_1 + c_{22}b_2 + \ldots + c_{n2}b_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c_{1i}b_1 + c_{2i}b_2 + \ldots + c_{ni}b_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c_{n1}b_1 + c_{n2}b_2 + \ldots + c_{nn}b_n \end{bmatrix}$

Hence, $x_i = \frac{1}{|A|}(c_{1i}b_1 + c_{2i}b_2 + ... + c_{ni}b_n)$ for i = 1, 2, ..., n.

i.e., $x_1 = \frac{1}{|A|}\sum_{j=1}^{n} b_i c_{ji}$ where i = 1, 2, ..., n.

From properties of cofactors we know that

$$\sum_{j=1}^{n} a_{ji}c_{ji} = |A| = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1i} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2i} & ... & a_{2n} \\ ... & ... & ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{ni} & ... & a_{nn} \end{bmatrix}$$

Hence replacing a_{ji} by b_j on both sides for j = 1, 2, ..., n; we get

$$\sum_{j=1}^{n} b_j c_{ji} = \begin{bmatrix} a_{11} & a_{12} & ... & b_{1i} & ... & a_{1n} \\ a_{21} & a_{22} & ... & b_{2i} & ... & a_{2n} \\ ... & ... & ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & b_{ni} & ... & a_{nn} \end{bmatrix}$$

or, $\sum_{j=1}^{n} b_j c_{ji} = |A_i|$,

where A_i is obtained from metrix A by replacing the i^{th} column by the

column $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

In otherwords, $x_i = \frac{1}{|A|}.|A_i| = \frac{|A_i|}{|A|}$

Hence the solution sof the system (1) are given by

$$x_1 = \frac{|A_1|}{|A|}, x_2 = \frac{|A_2|}{|A|}, ..., x_i = \frac{|A_i|}{|A|}, ..., x_n = \frac{|A_n|}{|A|}.$$

This method of solution is known as **Cramer's Rule**.

14.3.3 Illustative Examples

Example 1 : Solve by Cramer's rule :

$$x - y + 2z = 1$$
$$2x + y + z = 2$$
$$x - 3y + z = 1$$

Solution : $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 1 \end{bmatrix}$

$\therefore\ |A| = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 1 \end{bmatrix} = -9$

As $|A| \neq 0$, unique solution exists, where

$x = \frac{|A_1|}{|A|}, \ y = \frac{|A_2|}{|A|}, \ z = \frac{|A_3|}{|A|}$

Now $|A_1| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 1 \end{vmatrix} = -9$

$|A_2| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$

$|A_3| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{vmatrix} = 0$

Hence $x = \frac{-9}{-9} = 1,$

$y = \frac{0}{-9} = 0,$

$z = \frac{0}{-9} = 0.$

Example 2 : Solve by Cramer's rule :

$x_1 + x_2 - 2x_3 = 1$

$2x_1 - 7x_3 = 3$

$x_1 + x_2 - x_3 = 5$

Solution : $|A| = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 0 & -7 \\ 1 & 1 & -1 \end{vmatrix} = -2,$

$$|A_1| = \begin{vmatrix} 1 & 1 & -2 \\ 3 & 0 & -7 \\ 5 & 1 & -1 \end{vmatrix} = -31,$$

$$|A_2| = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 3 & -7 \\ 1 & 5 & -1 \end{vmatrix} = 13,$$

$$|A_3| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & 5 \end{vmatrix} = -8$$

The solutions are $x_1 = \dfrac{|A_1|}{|A|} = \dfrac{31}{2},$

$$x_2 = \frac{|A_2|}{|A|} = -\frac{13}{2},$$

$$x_3 = \frac{|A_3|}{|A|} = 4.$$

Example 3 : Solve by Cramer's rule :

$$x_1 + x_2 + x_3 + x_4 = 2$$
$$x_1 + 2x_2 + 3x_3 + 4x_4 = 2$$
$$2x_1 + 3x_2 + 5x_3 + 9x_4 = 2$$
$$x_1 + x_2 + 2x_3 + 7x_4 = 2$$

Solution : $|A| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 9 \\ 1 & 1 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 \\ 2 & 1 & 3 & 7 \\ 1 & 0 & 1 & 6 \end{vmatrix},$

by $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, $C_4 \to C_4 - C_1$

$$= 1.\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 7 \\ 0 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 1 & 6 \end{vmatrix}, \; R_2 \to R_2 - R_1$$

$$|A_1| = \begin{vmatrix} 2 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \\ 2 & 3 & 5 & 9 \\ 2 & 1 & 2 & 7 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 9 \\ 1 & 1 & 2 & 7 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 4 & 8 \\ 1 & 0 & 1 & 6 \end{vmatrix}$$

$$= 2.\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 0 & 1 & 6 \end{vmatrix} = 4\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 0 & 1 & 6 \end{vmatrix} = 4\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 6 \end{vmatrix}$$

$$= 4.(0-1) = -4$$

$$|A_2| = \begin{vmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 2 & 5 & 9 \\ 1 & 2 & 2 & 7 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 4 \\ 2 & 1 & 5 & 9 \\ 1 & 1 & 2 & 7 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 7 \\ 1 & 0 & 1 & 6 \end{vmatrix} = 2.\begin{vmatrix} 0 & 2 & 3 \\ -1 & 3 & 7 \\ 0 & 1 & 6 \end{vmatrix}$$

$$= 2.(-1)(-1)(12-3) = 18$$

$$|A_3| = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 2 & 4 \\ 2 & 3 & 2 & 9 \\ 1 & 1 & 2 & 7 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 4 \\ 2 & 3 & 1 & 9 \\ 1 & 1 & 1 & 7 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 7 \\ 1 & 0 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 3 \\ 1 & -1 & 7 \\ 0 & 0 & 6 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 6 \end{vmatrix} = 2(-6) = -12$$

$$|A_4| = \begin{vmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 2 \\ 1 & 1 & 2 & 2 \end{vmatrix} = 2.\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & 1 \\ 1 & 1 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 3 & -1 \\ 1 & 0 & 1 & 0 \end{vmatrix}$$

$$= 2.\begin{vmatrix} 1 & 2 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 0 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 2.(0+1) = 2$$

$$\therefore \quad x_1 = \frac{|A_1|}{|A|} = -\frac{4}{2} = -2,$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{18}{2} = 9,$$

$$x_3 = \frac{|A_3|}{|A|} = -\frac{12}{2} = -6,$$

$$x_4 = \frac{|A_4|}{|A|} = \frac{2}{2} = 1.$$

Example 4 : Investigate existence of solution of the system

$$2x_1 + 4x_2 - x_3 = 4$$
$$x_1 + 2x_2 - 2x_3 = 5$$
$$x_1 + 2x_2 - x_3 = -1$$

Solution : $|A| = \begin{vmatrix} 2 & 4 & -1 \\ 1 & 2 & -2 \\ 1 & 2 & -1 \end{vmatrix} = 2\begin{vmatrix} 2 & 2 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & -1 \end{vmatrix} = 2.0 = 0$

Since $|A| = 0$, Cramer's rule cannot be used to solve the system. In fact, the system is inconsistent and has no solution. Because, subtracting 3 times the third equation from the sum of the first two equations we get $0 = 12$, an absurdity.

CHECK YOUR PROGRESS

Q.1. Express in matrix form AX = B, examine uniqueness of solution and then solve calculating A^{-1} where ever unique solution exists :

i) $10x - 15y = 0$
$3x - 4y = 1$

ii) $3x + 2y - 2z = 1$
$-x + y + 4z = 1$
$2x - 3y + 4z = 8$

iii) $7x_1 + 6x_2 + 5x_3 = 1$
$x_1 + 2x_2 + x_3 = -1$
$3x_1 - 2x_2 + x_3 = 4$

Q.2. Solve by Cramer's rule :

i) $2x - y + 3z = 9$
$x + y + z = 6$
$x - y + z = 2$

ii) $2x_1 + x_2 + 5x_3 + x_4 = 5$
$x_1 + x_2 - 3x_3 - 4x_4 = -1$
$3x_1 + 6x_2 - 2x_3 + x_4 = 8$
$2x_1 + 2x_2 + 2x_3 - 3x_4 = 2$

14.4 A SYSTEM OF *m* NON-HOMOGENEOUS EQUATIONS IN *n* UNKNOWNS

A system of equations :

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ &... ... ... \quad .... (1) \\ &... ... ... \\ &a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{aligned}$$

where a_{ij}, $b_i \in$ R or any field F, is called a system of *m* non-homogeneous equations in *n* unknowns. The system can be written as :

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \ (i = 1, 2, ..., m)$$

$$\text{The matrix } A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & ... & ... & ... & a_{1n} \\ a_{21} & a_{22} & ... & ... & ... & a_{2n} \\ ... & ... & ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & ... & ... & a_{mn} \end{bmatrix}$$

is called the **co-efficient matrix**, and the matrix

$$(A|B) = (a_{ij}|b_i)_{m\times(n+1)} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} & b_1 \\ a_{21} & a_{22} & ... & a_{2n} & b_2 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} & b_m \end{bmatrix}$$

is called the **augmented matrix** of the system. Sometimes the last two columns of (A|B) are separated by a dotted line which is not obligatory.

An *n*-tuple $(\alpha_1, \alpha_2, ..., \alpha_n)$ which satisfies each of the *m* equations in (1) is called a **solution** of the equations.

14.4.1 Consistent and Inconsistent System

The system of equations :

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \text{ where } i = 1, 2, ..., m$$

is called a **consistent system** if it has at least one set of solution, otherwise the system is called **inconsistent**.

Example 1 : The system $x + 2y - 3z - 4t = 2$

$2x + 4y - 5z - 7t = 7$

$-3x - 6y + 11z + 14t = 0$

is a consistent system, since (3, 4, 3, 0) is a solution of the system.

Exampe 2 : The system $x_1 + 2x_2 - 3x_3 + 4x_4 = 2$

$x_2 + 4x_3 - 7x_4 = -3$

$2x_2 + 8x_3 - 14x_4 = 3$

is inconsistent, because subtracting twice the second equation from the third we shall get 0 = 9, an absurdity. So the system has no solution.

14.4.2 Equivalent Systems of Equations

The system of equations :

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \ (i = 1, 2, ..., m)$$

and $$\sum_{j=1}^{n} a'_{ij}x_j = b'_i, \ (i = 1, 2, ..., m)$$

are said to be **equivalent** if every solution of one system is also a solution of the other system.

Example : The systems of equations $x_1 - 2x_2 - 3x_3 = 4$

$2x_1 - 3x_2 + x_3 = 5$

and the system $x_1 + 11x_3 = -2$

$x_2 + 7x_3 = -3$

are equivalent. It can be verified that any set of solution of one system is also a solution of the other system. For example (9, 4, –1) is a solution of both the systems.

14.4.3 Homogeneous System of Equations

The system of equations :

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \ (i = 1, 2, ..., m)$$

is called a **homogeneous system** if $b_i = 0$ for i = 1, 2, ..., m. In other-words,

$$\sum_{j=1}^{n} a_{ij}x_j = 0, \ (i = 1, 2, ..., m)$$

is a homogeneous system.

Obviously, (0, 0, ..., 0) is a solution of any homogeneous system of equations. It may have other solutions also.

Example : $x_1 + 2x_2 - x_3 = 0$

$2x_1 + 5x_2 + 2x_3 = 0$

$x_1 + 4x_2 + 7x_3 = 0$

$x_1 + 3x_2 + 3x_3 = 0$ is a homogeneous system.

i) (0, 0, 0) is a solution of the system.

ii) (9, –4, 1) is another solution of the system.

14.4.4 Gaussian Elimination Method of Solution

The Gaussian elimination method of solving a system of linear equations is based on the following theorem, stated without proof.

Theorem : If $\sum_{j=1}^{n} a_{ij}x_j = b_i$, (i = 1, 2, ..., m) be a system of linear equations with augmented matrix $(A|B) = (a_{ij}|b_i)$ and $(R|S) = (\rho_{ij}|\delta_i)$ be its reduced echelon matrix, then the system

$\sum_{j=1}^{n} \rho_{ij}x_j = \delta_i$ (i = 1, 2, ..., n) is equivalent to the original system.

Thus, to solve a system of linear equations, its augmented matrix should be transformed into a reduced echelon matrix (or, row canonical form). Then the solution of the system determined by this reduced echelon matrix is also a solution to the original system.

GAUSS ELIMINATION METHOD FOR REDUCTION OF THE AUGMENTED MATRIX : The Gaussian elimination method for reducing the augmented matrix (A|B) to the row canonical form (R|S) is as follows :

Step 1 : First reduce (A|B) to an echelon form. If the echelon matrix has a row (0, 0, ..., 0, b) where b ≠ 0, then the system of equations is inconsitent, i.e., no solution exists. Otherwise, we follow step-2.

Step 2 : Let R_1, R_2, ..., R_r be the non-zero rows in the echelon matrix with leading entries α_{1j_1}, α_{2j_2}, ..., α_{rj_r} respectively. If $\alpha_{rj_r} \neq 1$, make it 1 by the elementary row operation

$R_r \to \frac{1}{\alpha_{rj_r}} R_r$. Now produce as above $\alpha_{rj_r} = 1$ by applying

row operations on R_1, R_2, ..., R_{r-1}.

Step 3 : Repeat step-2 in succession with R_{r-1}, R_{r-2}, ..., R_2. Finally,

if necessary, multiply R_1 by $\frac{1}{a_{1j_1}}$ to make $a_{1j_1} = 1$.

The matrix is row in row canonical form, from which we find the solution of the equivalent system, and hence the solution of the original system.

14.4.5 Illustrative Examples

Example 1 : Solve by Gaussian elimination method :

$$x - 2y + z = 7$$
$$2x - y + 4z = 17$$
$$3x - 2y + 2z = 14$$

Solution : We change the augmented matrix (A|B) to echelon form and then to row canonical form as follows :

$$(A|B) = \begin{bmatrix} 1 & -2 & 1 & 7 \\ 2 & -1 & 4 & 17 \\ 3 & -2 & 2 & 14 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 3 & 2 & 3 \\ 0 & 4 & -1 & -7 \end{bmatrix}, R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 3 & 2 & 3 \\ 0 & 0 & -\frac{11}{3} & -11 \end{bmatrix}, R_3 \to R_3 - \frac{4}{3}R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 3 & 2 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}, R_3 \to -\frac{3}{11}R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 1 & 3 \end{bmatrix}, R_1 \to R_1 - R_3,\ R_2 \to R_2 - 2R_3$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}, R_2 \to \frac{1}{3}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}, R_1 \to R_1 + 2R_2$$

which is in row canonical form. This give the equivalent system of equations

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \Rightarrow x = 2,\ y = -1,\ z = 3$$

Hence (2, –1, 3) is the unique solution of the given system of equations.

Example 2 : Solve the system :

$$x + 2y - 3z - 4t = 2$$
$$2x + 4y - 5z - 7t = 7$$
$$3x + 6y - 11z - 14t = 0$$

Solution : We reduce the augonented matrix to echelon form and then to row canonical form :

$$\begin{pmatrix} 1 & 2 & -3 & -4 & 2 \\ 2 & 4 & -5 & -7 & 7 \\ 3 & 6 & -11 & -14 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -3 & -4 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & -2 & -2 & -6 \end{pmatrix}, R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$$

$$\sim \begin{pmatrix} 1 & 2 & -3 & -4 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R_3 \to R_3 + 2R_2$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & -1 & 11 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, R_1 \to R_1 + 3R_2$$

which is in row canonical form. This equivalent system is

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $x + 2y - t = 11, z + t = 3$

$\Rightarrow$ $x = 11 - 2y + t, z = 3 - t$

Taking arbitrary values $y = \alpha$, $t = \beta$ we get

$x = 11 - 2\alpha + \beta$, $z = 3 - \beta$.

Hence, the system has **infinite number** of solutions given by $(11 - 2\alpha + \beta, \alpha, 3 - \beta, \beta)$ for arbitrarilly chosen values of α and β.

In particular, taking $\alpha = 1$, $\beta = 0$ we get a solution (9, 1, 3, 0).

Taking $\alpha = 0$, $\beta = 1$ we get another solution (12, 0, 2, 1), etc.

Example 3 : Solve the system :

$3x_1 + 4x_2 - x_3 + 2x_4 = 1$

$x_1 - 2x_2 + 3x_3 + x_4 = 2$

$3x_1 + 14x_2 - 11x_3 + x_4 = 3$

Solution : Augmented matrix :

$$= \begin{pmatrix} 3 & 4 & -1 & 2 & 1 \\ 1 & -2 & 3 & 1 & 2 \\ 3 & 14 & -11 & 1 & 3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 3 & 4 & -1 & 2 & 1 \\ 3 & 14 & -11 & 1 & 3 \end{pmatrix}, R_1 \leftrightarrow R_2$$

$$\sim \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 10 & -10 & -1 & -5 \\ 0 & 10 & -10 & -1 & 3 \end{pmatrix}, R_2 \to R_2 - 3R_1,\ R_3 \to R_3 - R_2$$

$$\sim \begin{pmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 10 & -10 & -1 & -5 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}, R_3 \to R_3 - R_2$$

It is in echelon form having the row (0, 0, 0, 0, 7). Hence the system has no solution.

[**Note :** In fact, the third row gives an equation

$0.x_1 + 0.x_2 + 0.x_3 + 0.x_4 = 7$

i.e., 0 = 7; hence the system is inconsistent.]

Example 4 : Find the conditions under which the following system has (i) no solution (ii) a solution :

$$x + 2y - 3z = a$$
$$2x + 6y - 11z = b$$
$$x - 2y + 7y = c$$

Solution : Augmented matrix

$$= \begin{pmatrix} 1 & 2 & -3 & a \\ 2 & 6 & -11 & b \\ 1 & -2 & 7 & c \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & -4 & 10 & c-a \end{pmatrix}, R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1$$

$$\sim \begin{pmatrix} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & 0 & 0 & 2b+c-5a \end{pmatrix}, R_3 \to R_3+2R_2$$

The equivalent system is

$$\begin{aligned} x + 2y - 3z &= 0 \\ 2y - 5z &= b - 2a \\ 0 &= 2b + c - 5a \end{aligned}$$

i) The system will have no solution if $2b + c - 5a \neq 0$.

ii) If $2b + c - 5a = 0$, i.e., $5a = 2b + c$, the system will have solutions given by the equivalent system.

$$\begin{aligned} x + 2y - 3z &= a \\ 2y - 5z &= b - 2a \\ \text{or, } x &= a - 2y + 3z, \\ y &= \frac{b-2a+5z}{2}. \end{aligned}$$

Taking arbitrarilly chosen values of z and applying the condition

$5a = 2b + c$, we shall get in finite number of solutions.

Note : This system cannot have a unique solution.

From the illustrative examples discussed above, it is clear that for a system of linear equations any one of the following holds :

i) has a unique solution,

ii) has no solution

iii) has an infinite number of solutions.

14.4.6 Solutions of the Homogeneous System

For the homogeneous system of linear equations :

$$\sum_{j=1}^{n} a_{ij}x_j = 0, \ (i = 1, 2, ..., m)$$

the last column of the augmented matrix has all zeros, and so, whatever elementary row operations are performed to reduce the augmented matrix to echelon form, the last column does not change. Hence we reduce the cofficient matrix $A = (a_{ij})_{m\times n}$ to canonical form to obtain the equivalent system. We explain the method in the following examples.

Example 1 : Solve the system : $x + 3y - 2z = 0$

$2x - y + 4z = 0$

$x - 11y + 14z = 0$

Solution : $A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{pmatrix}$

$$\sim \begin{pmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{pmatrix}, R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1$$

$$\sim \begin{pmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{pmatrix}, R_3 \to R_3 - 2R_2$$

$$\sim \begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & -\frac{8}{7} \\ 0 & 0 & 0 \end{pmatrix}, R_2 \to -\frac{1}{7}R_2$$

$$\sim \begin{pmatrix} 1 & 0 & \frac{10}{7} \\ 0 & 1 & -\frac{8}{7} \\ 0 & 0 & 0 \end{pmatrix}, \text{which is in row cononical form.}$$

So, the equivalent system is given by

$$\begin{pmatrix} 1 & 0 & \frac{10}{7} \\ 0 & 1 & -\frac{8}{7} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \ x + \frac{10}{7}z = 0, \ \ y - \frac{8}{7}y = 0$$

$$\Rightarrow \ x = -\frac{10}{7}z, \ \ y = \frac{8}{7}z.$$

Taking z = α, an arbitrary value, we get $(-\frac{10}{7}\alpha, \frac{8}{7}\alpha, \alpha)$ as the general solution of the system and hence the system has infinite number of solutions.

Example 2 : Solve the system : $x_1 + 2x_2 + x_3 - 3x_4 = 0$

$$2x_1 + 4x_2 + 3x_3 + x_4 = 0$$

$$3x_1 + 6x_2 + 4x_3 - 2x_4 = 0$$

Solution : Coefficient matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & -3 \\ 2 & 4 & 3 & 1 \\ 3 & 6 & 4 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 4 & 7 \end{pmatrix}, R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & -10 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}, R_1 \to R_1 - R_2, R_3 \to R_3 - R_2$$

Hence, the system is equivalent to $x_1 + 2x_2 - 10x_4 = 0$

$$x_3 + 7x_4 = 0$$

Setting $x_2 = a$, $x_4 = b$,

we get the general solution as (10b – 2a, a, –7b, b)

where a, b are arbitrary. The system has infinite number of solutions.

Example 3 : Find a non-zero solution of the system :

$$x_1 - x_2 + x_3 = 0$$

$$x_1 + x_2 + 2x_3 = 0$$

$$x_1 + 2x_2 - x_3 = 0$$

Solution : The coefficient matrix

$$= \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 3 & -2 \end{pmatrix}, R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$$

$$\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 3 & -2 \end{pmatrix}, R_2 \to ½R_2$$

$$\sim \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -\frac{7}{2} \end{pmatrix}, R_1 \to R_1 + R_2,\ R_3 \to R_3 - 3R_1$$

$$\sim \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}, R_3 \to \frac{2}{7}R_3$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_1 \to R_1 - \frac{3}{2}R_3,\ R_2 \to R_2 - ½R_3$$

Hence the system is equivalent to $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

In otherwords, the system has unique solution (0, 0, 0) and it has no non-zero solution.

Note : A system of homogeneous linear equations has either (i) a unique solution (0, 0, 0), called the **trivial solution** or (ii) an infinite number of solutions.

CHECK YOUR PROGRESS

Q.3. Solve by using Gaussian elimination method :

$$x + 2y - z = 3$$
$$3x - y + 2z = 1$$
$$2x - 2y + 3z = 2$$
$$x - y + z = -1$$

Q.4. Examine existence of solution of the system

$$x + y - 2z + 3t = 4$$
$$2x + 3y + 3z - t = 3$$
$$5x + 7y + 4z + t = 5$$

Q.5. For what value of a the following system is consistent? Find solutions when consistent.

$$x - y + z = 1$$
$$x + 2y + 4z = a$$
$$x + 4 + 6z = a^2.$$

14.5 LET US SUM UP

- $\sum_{j=1}^{n} a_{ij}x_j = b_i$, where i = 1, 2, ..., n is a system of *n* linear non-homogeneous equations in *n* unknowns. In matrix form the system can be expressed as AX = B,

  where $A = [a_{ij}]_{n\times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

- The system AX = B has a unique solution when $|A| \neq 0$.
- By Cramer's rule, the solution of the system is given by $x_i = \frac{|A_i|}{|A|}$ for i = 1, 2, ..., n, where A_i is obtained from A replacing the i^{th} column by the column $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.
- The system of equations $\sum_{j=1}^{n} a_{ij}x_j = b_i$, where i = 1, 2, ..., m is called a system of a *m* linear non-homogeneous equations in *n* unknowns where $A = [a_{ij}]_{m\times n}$ is called the **Co-efficient matrix** and (A|B) = $[a_{ij}|b_i]_{m\times n}$ is called the **Augmented matrix**.
- The system is called **consistent** if it has atleast one soloution i.e., atleast one n-tuple $(\alpha_1, \alpha_2, ..., \alpha_n)$ which satisfies the equations in the system, otherwise the system is called inconsistent.
- Two systems $\sum_{j=1}^{n} a_{ij}x_j = b_i$ (i = 1, 2, ..., m) and $\sum_{j=1}^{n} a'_{ij}x_j = b'_i$ (i = 1, 2, ..., m) are said to be equivalent if every solution of one system is also a solution of the other system.

- The system can be solved by reducing the augmented matrix to echelon form and then to row canonical form using Gaussian Elimination Method; thereby finding solution of an equivalent system and hence that of the original system.
- Any one of the following three holds for the system :
  - i) has a unique solution
  - ii) has no solution
  - iii) has an infinite number of solutions.
- The system of equations $\sum_{j=1}^{n} a_{ij}x_j = 0$, where i = 1, 2, ..., m is called a homogeneous system. The system is solved by reducing the co-efficient matrix to row canonical form.
- A homogeneous system has either (i) a unique solution (0, 0, 0), called the trivial solution or (ii) an infinite number of solution.

14.6 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 :i) Matrix form is $\begin{pmatrix} 10 & -5 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

or, AX = B, where $A = \begin{pmatrix} 10 & -15 \\ 3 & -4 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$|A| = \begin{vmatrix} 10 & -15 \\ 3 & -4 \end{vmatrix} = -40 + 45 = 5 \neq 0$$

So, unique solution exists and is given by $X = A^{-1}B$.

$$\text{Now, } A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{5}\begin{pmatrix} -4 & -3 \\ 15 & 10 \end{pmatrix}' = \frac{1}{5}\begin{pmatrix} -4 & 15 \\ -3 & 10 \end{pmatrix}$$

$$\Rightarrow X = \frac{1}{5}\begin{pmatrix} -4 & 15 \\ -3 & 10 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 15 \\ 10 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} \\ 2 \end{pmatrix} \Rightarrow x = 3,\ \ y = 2.$$

ii) Matrix form is AX = B,

where $A = \begin{pmatrix} 3 & 2 & -2 \\ -1 & 1 & 4 \\ 2 & -3 & 4 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \\ 8 \end{pmatrix}$

$$|A| = \begin{vmatrix} 3 & 2 & -2 \\ -1 & 1 & 4 \\ 2 & -3 & 4 \end{vmatrix} = 70,\ \text{adj } A = \begin{pmatrix} 16 & -2 & 10 \\ 12 & 16 & -10 \\ 1 & 13 & 5 \end{pmatrix}$$

Hence, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{70}\begin{pmatrix} 16 & -2 & 10 \\ 12 & 16 & -10 \\ 1 & 13 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 8 \end{pmatrix}$

$$= \frac{1}{70}\begin{pmatrix} 94 \\ -52 \\ 54 \end{pmatrix} = \begin{pmatrix} \frac{47}{35} \\ -\frac{26}{35} \\ \frac{27}{35} \end{pmatrix}$$

$$\Rightarrow x = \frac{47}{35},\ y = -\frac{26}{35},\ z = \frac{27}{35}.$$

iii) AX = B, where $A = \begin{pmatrix} 7 & 6 & 5 \\ 1 & 2 & 1 \\ 3 & -2 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$.

$$|A| = \begin{vmatrix} 7 & 6 & 5 \\ 1 & 2 & 1 \\ 3 & -2 & 1 \end{vmatrix} = 0 \Rightarrow \text{the solution does not exist.}$$

Ans. to Q. No. 2 : i) $|A| = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = -2\ |A_1| = \begin{vmatrix} 9 & -1 & 3 \\ 6 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = -2$

$$|A_2| = \begin{vmatrix} 2 & 9 & 3 \\ 1 & 6 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -4 \qquad |A_3| = \begin{vmatrix} 2 & -1 & 9 \\ 1 & 1 & 6 \\ 1 & -1 & 2 \end{vmatrix} = -6$$

So, the solutions are $x = \dfrac{|A_1|}{|A|} = \dfrac{-2}{-2} = 1,$

$$y = \frac{|A_2|}{|A|} = \frac{-4}{-2} = 2,$$

$$z = \frac{|A_3|}{|A|} = \frac{-6}{-2} = 3.$$

ii) $$|A| = \begin{vmatrix} 2 & 1 & 5 & 1 \\ 1 & 1 & -3 & -4 \\ 3 & 6 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & -3 & -4 \\ 2 & 1 & 5 & 1 \\ 3 & 6 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix}, \ R_1 \leftrightarrow R_2$$

$$= -\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 11 & 9 \\ 3 & 3 & 7 & 13 \\ 2 & 0 & 8 & 5 \end{vmatrix}; \ c_2 \to c_2 - c_1,\ c_3 \to c_3 + 3c_1,\ c_4 \to c_4 + 4c_1$$

$$= -1.\begin{vmatrix} -1 & 11 & 9 \\ 3 & 7 & 13 \\ 0 & 8 & 5 \end{vmatrix} = -120$$

$$|A_1| = \begin{vmatrix} 5 & 1 & 5 & 1 \\ -1 & 1 & -3 & -4 \\ 8 & 6 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix} = -1\begin{vmatrix} -1 & 1 & -3 & -4 \\ 5 & 1 & 5 & 1 \\ 8 & 6 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix}, \ R_1 \leftrightarrow R_2$$

$$= \begin{vmatrix} 1 & -1 & 3 & 4 \\ 5 & 1 & 5 & 1 \\ 8 & 6 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 5 & 6 & -10 & -19 \\ 8 & 14 & -26 & -31 \\ 2 & 4 & -4 & -11 \end{vmatrix};$$

$$c_2 \to c_2 + c_1,\ c_3 \to c_3 - 3c_1,\ c_4 \to c_4 - 4c_1$$

$$= 1.\begin{vmatrix} 6 & -10 & -19 \\ 14 & -26 & -31 \\ 4 & -4 & -11 \end{vmatrix} = \begin{vmatrix} 6 & 10 & 19 \\ 14 & 26 & 31 \\ 4 & 4 & 11 \end{vmatrix}$$

$$= 4.\begin{vmatrix} 3 & 5 & 19 \\ 7 & 13 & 31 \\ 2 & 2 & 11 \end{vmatrix} = -240$$

$$|A_2| = \begin{vmatrix} 2 & 5 & 5 & 1 \\ 1 & -1 & -3 & -4 \\ 3 & 8 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix} = -\begin{vmatrix} 1 & -1 & -3 & -4 \\ 2 & 5 & 5 & 1 \\ 3 & 8 & -2 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 11 & 9 \\ 3 & 11 & 7 & 13 \\ 2 & 4 & 8 & 5 \end{vmatrix} = -1.\begin{vmatrix} 7 & 11 & 9 \\ 11 & 7 & 13 \\ 4 & 8 & 5 \end{vmatrix} = -24$$

$$|A_3| = \begin{vmatrix} 2 & 1 & 5 & 1 \\ 1 & 1 & -1 & -4 \\ 3 & 6 & 8 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & -1 & -4 \\ 2 & 1 & 5 & 1 \\ 3 & 6 & 8 & 1 \\ 2 & 2 & 2 & -3 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 7 & 9 \\ 3 & 3 & 11 & 13 \\ 2 & 0 & 4 & 5 \end{vmatrix} = -1.\begin{vmatrix} -1 & 7 & 9 \\ 3 & 11 & 13 \\ 0 & 4 & 5 \end{vmatrix} = 0$$

$$|A_4| = \begin{vmatrix} 2 & 1 & 5 & 5 \\ 1 & 1 & -3 & -1 \\ 3 & 6 & -2 & 8 \\ 2 & 2 & 2 & 2 \end{vmatrix} = -\begin{vmatrix} 2 & 2 & 2 & 2 \\ 2 & 1 & 5 & 5 \\ 1 & 1 & -3 & -1 \\ 3 & 6 & -2 & 8 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 5 & 5 \\ 1 & 1 & -3 & -1 \\ 3 & 6 & -2 & 8 \end{vmatrix} = -2\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 3 & 3 \\ 1 & 0 & -4 & -2 \\ 3 & 6 & -5 & 5 \end{vmatrix}$$

$$= -2.\begin{vmatrix} -1 & 3 & 3 \\ 0 & -4 & -2 \\ 3 & -5 & 5 \end{vmatrix} = -96$$

$\therefore$ The solutions are

$$x_1 = \frac{|A_1|}{|A|} = \frac{-240}{-120} = 2, \qquad x_2 = \frac{|A_2|}{|A|} = \frac{-24}{-120} = \frac{1}{5},$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{0}{-120} = 0, \qquad x_4 = \frac{|A_4|}{|A|} = \frac{-96}{-120} = \frac{4}{5}.$$

Ans. to Q. No. 3 : Augmented matrix

$$= \begin{pmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{pmatrix}; R_2 \to R_2-3R_1,\ R_3 \to R_3-2R_1,\ R_4 \to R_4-R_1$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & \frac{5}{7} & \frac{20}{7} \\ 0 & 0 & -\frac{1}{7} & -\frac{4}{7} \end{pmatrix}, R_3 \to R_3-\frac{6}{7}R_2,\ R_4 \to R_4-\frac{3}{7}R_2$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & \frac{5}{7} & \frac{20}{7} \\ 0 & 0 & 0 & 0 \end{pmatrix}, R_4 \to R_4+\frac{1}{5}R_3$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & -\frac{5}{7} & \frac{8}{7} \\ 0 & 0 & \frac{5}{7} & \frac{20}{7} \\ 0 & 0 & 0 & 0 \end{pmatrix}, R_2 \to -\frac{1}{7}R_2$$

$$\sim \begin{pmatrix} 1 & 0 & \frac{3}{7} & \frac{5}{7} \\ 0 & 1 & -\frac{5}{7} & \frac{8}{7} \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}, R_1 \to R_1 - 2R_2,\ R_3 \to \frac{7}{5}R_3$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ The equivalent system is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 4 \\ 0 \end{pmatrix}$

$\Rightarrow x = -1,\ y = 4,\ z = 4.$

Ans. to Q. No. 4 : Augmented matrix

$$= \begin{pmatrix} 1 & 1 & -2 & 3 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & 1 & 5 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 2 & 14 & -14 & -15 \end{pmatrix}, R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 5R_1$$

$$\sim \begin{pmatrix} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 0 & 0 & 0 & -5 \end{pmatrix}, R_3 \to R_3 - 2R_2$$

It is now in echelon form having a row (0, 0, 0, 0, –5). Hence the system is inconsistent, i.e., it has no solution.

Ans. to Q. No. 5 : Augmented matrix

$$= \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 2 & 4 & a \\ 1 & 4 & 6 & a^2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 3 & 3 & a-1 \\ 1 & 5 & 5 & a^2-1 \end{pmatrix}, R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$$

$$\sim \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & \frac{a-1}{3} \\ 0 & 5 & 5 & a^2-1 \end{pmatrix}, R_2 \to \frac{1}{3}R_2$$

$$\sim \begin{pmatrix} 1 & 0 & 2 & \frac{a+2}{3} \\ 0 & 1 & 1 & \frac{a-1}{3} \\ 0 & 0 & 0 & \frac{3a^2-5a+2}{3} \end{pmatrix}, R_1 \to R_1 + R_2,\ R_3 \to R_3 - 5R_1$$

The equivalent system is $x + 2z = \frac{a+2}{3}$,

$$y + z = \frac{a-1}{3}$$

$$0 = \frac{a^2-5a+2}{3},$$

The system is

i) inconsistent if $3a^2 - 5a + 2 \neq 0$,

ii) consistent if $\frac{3a^2-5a+2}{3} = 0$ i.e., $a = 1, \frac{2}{3}$

Case 1 : When $a = 1$, we get $x + 2z = 1$, $y + z = 0$ and so, taking $z = \alpha$ (arbitrary), we get infinite solution given by $(1-2\alpha, -\alpha, \alpha)$ for arbitrary values of α.

Case 2 : When $a = \frac{2}{3}$, we get $x + z = \frac{8}{9}$, $y + z = -\frac{1}{9}$.

Taking $z = \beta$ (arbitrary), we get another set of infinite solutions given by $(\frac{8}{9} - 2\beta, -\frac{1}{9} - \beta, \beta)$.

14.7 FURTHER READINGS

1. *Matrices* – J. N. Sharma & S. N. Goel.
2. *Linear Algebra*, Seymour Lipschutz.

13.8 MODEL QUESTIONS

Q.1. Solve by Cramer's rule :

i) $x - y + 2z = 1$
$2x + y + z = 2$
$x - 3y + z = 1$

ii) $x + y - 2z = 1$
$2x - 7z = 3$
$x + y - z = 5$

iii) $x + 2y - z = 3$
$3x - y + 2z = 1$
$2x - 2y + 3z = 2$

iv) $3x + 2y - 2z = 1$
$-x + y + 4z = 13$
$2x - 3y + 4z = 8$

Q.2. Solve by using Gaussian elimination method :

i) $x - 4y - 3z = -16$
$2x + 7y + 12z = 48$
$4x - y + 6z = 16$
$5x - 5y + 3z = 0$

ii) $x - 2y + z = 7$
$2x - y + 4z = 17$
$3x - 2y + 2z = 14$

iii) $x_1 + 2x_2 - 3x_3 = 1$
$x_2 - 2x_3 = 2$
$2x_2 - 4x_3 = 4$

iv) $2x_1 - 5x_2 + 3x_3 - 4x_4 + 2x_5 = 4$
$3x_1 - 7x_2 + 2x_3 - 5x_4 + 4x_5 = 9$
$5x_1 - 10x_2 - 5x_3 - 4x_4 + 7x_5 = 22.$

Q.3. Show that the following equations are inconsistent. Apply Gaussian elimination on the augmented matrix to show inconsistency.

i) $x_1 + 2x_2 - 3x_3 + 4x_4 = 2$
$x_2 + 4x_3 - 7x_4 = -3$
$2x_2 + 8x_3 - 14x_4 = 3$

ii) $x + 2y - 3z = 0$
$2x + 4y - 2y = z$
$3x + 6y - 4z = 3$

iii) $x + 2y - 3z = 1$
$2x + 6y - 11z = -1$
$x - 2y + 7z = 8$

Q.4. Determine the values of λ so that the following system has

i) a unique solution,

ii) no solution,

iii) infinite number of solutions.

a) $x + y - z = 1$

$2x + 3y + \lambda z = 3$

$x + \lambda y + 3z = 2$

b) $x + 2y + \lambda z = 0$

$2x + 3y - 2z = \lambda$

$\lambda x + y + \lambda^2 z = 3$

Q.5. Find solutions of the following homogeneous systems :

i) $x + 2y - 3z = 0$

$2x + 5y + 2z = 0$

$3x - y - 4z = 0$

ii) $x_1 + 2x_2 + 5x_3 + 2x_4 = 0$

$2x_1 + 4x_2 + x_3 - 5x_4 = 0$

$x_1 + 2x_2 + x_3 - 2x_4 = 0$

iii) $2x_1 + 4x_2 - 5x_3 + 3x_4 = 0$

$3x_1 + 6x_2 - 7x_3 + 4x_4 = 0$

$5x_1 + 10x_2 - 11x_3 + 6x_4 = 0$

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સ્વાધ્યાયઃ પરમં તપઃ

શિક્ષણ, સંસ્કૃતિ, સદ્‌ભાવ, દિવ્યબોધનું ધામ
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